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ORBIT EQUIVALENCE OF ERGODIC GROUP ACTIONS

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1. Group actions, equivalence relations and orbit equivalence

1.1. **Standard spaces.** A *Borel space* $(X, \mathcal{B}_X)$ is a pair consisting of a topological space X and its σ-algebra of Borel sets $\mathcal{B}_X$. If the topology on X can be chosen to the *Polish* (i.e., separable, metrizable and complete), then $(X, \mathcal{B}_X)$ is called a *standard Borel space.* For simplicity, we will omit the σ-algebra and write X instead of $(X, \mathcal{B}_X)$. If $(X, \mathcal{B}_X)$ and $(Y, \mathcal{B}_Y)$ are Borel spaces, then a map $\theta : X \to Y$ is called a *Borel isomorphism* if it is a bijection and θ and θ^{-1} are Borel measurable. Note that a Borel bijection θ is automatically a Borel isomorphism (see [Ke95, Corollary 15.2]).

Definition 1.1. A *standard probability space* is a measure space $(X, \mathcal{B}_X, \mu)$, where $(X, \mathcal{B}_X)$ is a standard Borel space and $\mu : \mathcal{B}_X \to [0,1]$ is a σ-additive measure with $\mu(X) = 1$. For simplicity, we will write (X, μ) instead of $(X, \mathcal{B}_X, \mu)$. If (X, μ) and (Y, ν) are standard probability spaces, a map $\theta : X \to Y$ is called a *measure preserving (m.p.) isomorphism* if it is a Borel isomorphism and measure preserving: $\theta_*\mu = \nu$, where $\theta_*\mu(A) = \mu(\theta^{-1}(A))$, for any $A \in \mathcal{B}_Y$.

Example 1.2. The following are standard probability spaces:

(1) $([0,1], \lambda)$, where $[0,1]$ is endowed with the Euclidean topology and its Lebegue measure λ.
(2) (G, m_G), where G is a compact metrizable group and m_G is the Haar measure of G (see [Ha74, Chapter XII] and [Fo99, Chapter 11]).
(3) Let (X_n, μ_n), $n \in \mathbb{N}$, be a sequence of standard probability spaces. Endow $X := \prod_{n\in\mathbb{N}} X_n$ with the product topology and the infinite product probability measure $\mu := \otimes_{n\in\mathbb{N}}\mu_n$. This satisfies $\mu(\prod_{n\in\mathbb{N}} A_n) = \prod_{n\in\mathbb{N}} \mu_n(A_n)$, for any sequence of Borel sets $A_n \subset X_n$, $n \in \mathbb{N}$ (see [Ha74, Chapter VII]). Then (X, μ) is a standard probability space.

For the following results, see [Ke95, Theorems 15.6 and 17.4].

Theorem 1.3 (the isomorphism theorems)**.** *The following hold:*

(1) Any uncountable standard Borel space X is Borel isomorphic to $[0,1]$.
(2) Any non-atomic standard probability space (X, μ) is isomorphic to $([0,1], \lambda)$.

Convention 1.4. Hereafter, null sets are neglected and equality is understood almost everywhere.

1.2. **Group actions.** Unless specified otherwise, all groups Γ that we consider are countable, discrete and infinite, and all probability spaces (X, μ) are standard. An action of a group Γ on a set X is a map $\Gamma \times X \to X$, which we denote by $(g, x) \mapsto g \cdot x$, such that $g \cdot (hx) = (gh) \cdot x$.

Definition 1.5. An action of a group Γ on a probability space (X, μ) is called *p.m.p.* (*probability measure preserving*) if for all $g \in \Gamma$, the map $x \mapsto g \cdot x$ is an m.p. isomorphism of X, i.e., a Borel isomorphism with $\mu(g \cdot A) = \mu(A)$, for all $A \in \mathcal{B}_X$. We denote p.m.p. actions by $\Gamma \curvearrowright (X, \mu)$.

Example 1.6. The following actions are p.m.p.:

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(1) If G is a compact metrizable group, then any subgroup $\Gamma < G$ acts on G by *left translation*:
$$g \cdot x = gx, \qquad \text{for every } g \in \Gamma \text{ and } x \in G.$$
This action preserves the Haar measure, m_G, and thus $\Gamma \curvearrowright (G, \mathrm{m}_G)$ is a p.m.p. action.

(2) If Γ is a countable group and (X, μ) a standard probability space, then Γ acts on X^Γ by
$$g \cdot x = (x_{g^{-1}h})_{h\in\Gamma}, \qquad \text{for every } g \in \Gamma \text{ and } x = (x_h)_{h\in\Gamma} \in X^\Gamma.$$
This action preserves the product measure $\mu^{\otimes\Gamma}$, and the p.m.p. action $\Gamma \curvearrowright (X^\Gamma, \mu^{\otimes\Gamma})$ is called a *Bernoulli action*.

(3) The standard (matrix multiplication) action of $\mathrm{SL}_n(\mathbb{Z})$ on the n-torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ endowed with the Lebesgue measure, for $n \geq 2$.

(4) If G is a Lie group and $\Gamma, \Lambda < G$ are lattices (e.g., $G = \mathrm{SL}_n(\mathbb{R})$, $\Gamma = \Lambda = \mathrm{SL}_n(\mathbb{Z})$, $n \geq 2$) then the left translation action of Γ on G/Λ given by $g\cdot(x\Lambda) = gx\Lambda$ preserves the probability measure $\mathrm{m}_{G/\Lambda}$ obtained by pushing forward the Haar measure m_G through $G \to G/\Lambda$.

1.3. **Equivalence relations.** Let X be a standard Borel space. A *partial isomorphism* is a Borel isomorphism $\theta : A = \mathrm{dom}(\theta) \to B = \mathrm{im}(\theta)$ between two Borel subsets A and B of X.

An equivalence relation $\mathcal{R}$ on X is called *Borel* if $\mathcal{R}$ is a Borel subset of $X \times X$, and *countable* if the $\mathcal{R}$-class $[x]_\mathcal{R} = \{y \in X \mid (x, y) \in \mathcal{R}\}$ is countable, for any $x \in X$. We denote by

- $[[\mathcal{R}]]$ the set, called the *full groupoid of* $\mathcal{R}$, of partial isomorphisms $\theta : A \to B$ such that $(x, \theta(x)) \in \mathcal{R}$, for all $x \in A$, and by
- $[\mathcal{R}]$ the group, called the *full group of* $\mathcal{R}$, of Borel isomorphisms $\theta : X \to X$ such that $(x, \theta(x)) \in \mathcal{R}$, for all $x \in X$.

Definition 1.7. Let (X, μ) be a standard probability space. An equivalence relation $\mathcal{R}$ on (X, μ) is called *countable p.m.p.* (or simply, *p.m.p.*) if it is countable, Borel and every partial isomorphism $\theta : A \to B$ belonging to $[[\mathcal{R}]]$ is measure preserving: $\mu(\theta(C)) = \mu(C)$, for every Borel set $C \subset A$.

Remark 1.8. Let $\mathcal{R}$ be a p.m.p. equivalence relation on (X, μ). Let $\theta_0 \in [[\mathcal{R}]]$ and denote $A_0 = \mathrm{dom}(\theta_0)$ and $B_0 = \mathrm{im}(\theta_0)$. Assume that $\mu(A_0) > 0$ and let $A \supset A_0, B \supset B_0$ be Borel sets such that $\mu(A \setminus A_0) = \mu(B \setminus B_0) = 0$. Then we can find a partial isomorphism $\theta : A \to B$ such that $\theta(x) = \theta_0(x)$, for almost every $x \in A_0$. Let $N \subset A_0$ be an uncountable null Borel set. Then $(A \setminus A_0) \cup N$ and $(B \setminus B_0) \cup \theta_0(N)$ are uncountable standard Borel spaces. By Theorem 1.3 there is a Borel isomorphism $\theta' : (A \setminus A_0) \cup N \to (B \setminus B_0) \cup \theta_0(N)$. Then $\theta : A \to B$ defined by $\theta_{|A_0\setminus N} = \theta_{0|A_0\setminus N}$ and $\theta_{|(A\setminus A_0)\cup N} = \theta'$ satisfies the desired conditions.

Convention: Remark 1.8 allows us to make the following convention: hereafter, we write that a partial isomorphism $\theta : A \to B$ belongs to $[[\mathcal{R}]]$ if there is a Borel co-null subset $A_0 \subset A$ such that $\theta_{|A_0}$ belongs to $[[\mathcal{R}]]$, in the above sense.

Exercise 1.9. Let $\Gamma \curvearrowright (X, \mu)$ be a p.m.p. action of a countable group on a standard probability space. Prove that the *orbit equivalence relation* $\mathcal{R}(\Gamma \curvearrowright X) := \{(x, y) \in X \times X \mid \Gamma \cdot x = \Gamma \cdot y\}$ is a countable p.m.p. equivalence relation.

Theorem 1.10 (Feldman-Moore, [FM77]). *Any countable p.m.p. equivalence relation $\mathcal{R}$ is the orbit equivalence relation, $\mathcal{R}(\Gamma \curvearrowright X)$, of a p.m.p. action of a countable group Γ.*

For a proof of this theorem, see [FM77] or [KM04, Chapter I].

Remark 1.11. A p.m.p. action $\Gamma \curvearrowright (X, \mu)$ is called *free* if every non-trivial element $g \in \Gamma \setminus \{e\}$ acts freely, in the sense that $\mu(\{x \in X \mid g \cdot x = x\}) = 0$. Furman [Fu98] found the first examples of countable p.m.p. equivalence relations which are not orbit equivalence relation of any free action.

1.4. **Ergodicity.** A p.m.p. action $\Gamma \curvearrowright (X,\mu)$ is called *ergodic* if any Γ-invariant Borel subset $Y \subset X$ is null or co-null, i.e., $\mu(Y) \in \{0,1\}$. A p.m.p. equivalence relation $\mathcal{R}$ on (X,μ) is called *ergodic* if any $\mathcal{R}$-invariant Borel subset $Y \subset X$ is null or co-null.

Remark 1.12. A p.m.p. action $\Gamma \curvearrowright (X,\mu)$ is ergodic iff $\mathcal{R}(\Gamma \curvearrowright X)$ is ergodic.

Exercise 1.13. Let $\Gamma \curvearrowright (X,\mu)$ be an ergodic p.m.p. action of a countable group Γ on a standard probability space (X,μ). Let $Y \subset X$ be a Borel set such that $\mu(g\cdot Y \triangle Y) = 0$, for every $g \in \Gamma$. Prove that Y is null or co-null.

Exercise 1.14. Let Γ be a countable dense subgroup of a compact metrizable group G. Prove that the left translation action $\Gamma \curvearrowright (G, \mathrm{m}_G)$ is free and ergodic.

Exercise 1.15. Let Γ be a countable group and (X,μ) be a standard probability space with more than one point. Prove that the Bernoulli action $\Gamma \curvearrowright (X^\Gamma, \mu^{\otimes\Gamma})$ is free and ergodic. Moreover, prove that this action is *mixing*: $\lim_{g\to\infty} \mu(g\cdot A \cap B) = \mu(A)\mu(B)$, for any Borel sets $A, B \subset X^\Gamma$.

Lemma 1.16. *Let $\mathcal{R}$ be an ergodic p.m.p. equivalence relation on (X,μ). Let $A, B \subset X$ be Borel sets with $\mu(A) = \mu(B) > 0$. Then there exists $\theta \in [\mathcal{R}]$ such that $\theta(A) = B$.*

Proof. We first prove that there is $\theta' \in [[\mathcal{R}]]$ such that $\mathrm{dom}(\theta') = A$ and $\mathrm{im}(\theta') = B$. Since $\mu(X\setminus A) = \mu(X\setminus B)$, we can thus also find $\theta'' \in [[\mathcal{R}]]$ such that $\mathrm{dom}(\theta'') = X\setminus A$ and $\mathrm{im}(\theta'') = X\setminus B$. Then $\theta : X \to X$ given by $\theta_{|A} = \theta'$ and $\theta_{|X\setminus A} = \theta''$ defines an element of $[\mathcal{R}]$ such that $\theta(A) = B$.

To prove the above assertion, we first apply Theorem 1.10 and Remark 1.12 to find an ergodic p.m.p. action $\Gamma \curvearrowright (X,\mu)$ of a countable group Γ such that $\mathcal{R} = \mathcal{R}(\Gamma \curvearrowright X)$. We define $\mathcal{F}$ to be the set of subsets $\mathcal{G}$ of $[[\mathcal{R}]]$ with the property that $\{\mathrm{dom}(\theta)\}_{\theta\in\mathcal{G}}$ are pairwise disjoint non-null subsets of A and $\{\mathrm{im}(\theta)\}_{\theta\in\mathcal{G}}$ are pairwise disjoint non-null subsets of B.

We claim that $\mathcal{F}$ is non-empty. Since Γ acts ergodically, the Γ-invariant non-null set $\cup_{g\in\Gamma}(g\cdot A)$ must be co-null in X. Thus, $\mu(\cup_{g\in\Gamma}(g\cdot A\cap B)) = \mu(B) > 0$, hence $\mu(g\cdot A\cap B) > 0$, for some $g \in \Gamma$. Define $\rho : A \cap g^{-1}\cdot B \to g\cdot A\cap B$ by letting $\rho(x) = g\cdot x$. Then $\rho \in [[\mathcal{R}]]$ and since $\mathrm{dom}(\rho) \subset A$ and $\mathrm{im}(\rho) \subset B$ are non-null sets, we get that $\{\rho\} \in \mathcal{F}$, and thus $\mathcal{F} \neq \emptyset$.

Next, we order $\mathcal{F}$ by inclusion: $\mathcal{G} \leq \mathcal{G}'$ iff $\mathcal{G} \subset \mathcal{G}'$. If $\{\mathcal{G}_i\}_{i\in I} \subset \mathcal{F}$ is a totally ordered family (i.e., such that $\mathcal{G}_i \leq \mathcal{G}_j$ or $\mathcal{G}_j \leq \mathcal{G}_i$, for every $i, j \in I$), then $\cup_{i\in I}\mathcal{G}_i \in \mathcal{F}$ is an upper bound for $\{\mathcal{G}_i\}_{i\in I}$. By Zorn's lemma we conclude that there exists a maximal element $\mathcal{G} \in \mathcal{F}$.

Since $\{\mathrm{dom}(\theta)\}_{\theta\in\mathcal{G}}$ are pairwise disjoint non-null Borel subsets of A and $\mu(A) \leq 1$, it follows that $\mathcal{G}$ is countable. Enumerate $\mathcal{G} = \{\theta_n\}_{n=1}^N$, where $N \in \mathbb{N}\cup\{\infty\}$, and denote $A_n = \mathrm{dom}(\theta_n)$, $B_n = \mathrm{im}(\theta_n)$, for every n. Define $A_0 = \cup_{n=1}^N A_n$ and $B_0 = \cup_{n=1}^N B_n$. Since $\theta_n \in [[\mathcal{R}]]$ and $\theta_n(A_n) = B_n$, we get that $\mu(A_0) = \sum_{n=1}^N \mu(A_n) = \sum_{n=1}^N \mu(B_n) = \mu(B_0)$, and thus $\mu(A \setminus A_0) = \mu(B \setminus B_0)$.

We claim that $\mu(A \setminus A_0) = \mu(B \setminus B_0) = 0$. Otherwise, if $\mu(A \setminus A_0) = \mu(B \setminus B_0) > 0$, by repeating the argument from above, we can find $\rho \in [[\mathcal{R}]]$ such that $\mathrm{dom}(\rho) \subset A \setminus A_0$ and $\mathrm{im}(\rho) \subset B \setminus B_0$ are non-null sets. But then it is clear that $\mathcal{G} \cup \{\rho\} \in \mathcal{F}$, which contradicts the maximality of $\mathcal{G}$.

Finally, we define $\theta' : A_0 \to B_0$ by letting $\theta'(x) = \theta_n(x)$, if $x \in A_n$. Then $\theta' \in [[\mathcal{R}]]$ and $\mathrm{dom}(\theta') = A_0$ and $\mathrm{im}(\theta') = B_0$. By Remark 1.8 this implies the assertion, and the conclusion. ■

Exercise 1.17. Let $\mathcal{R}$ be a countable p.m.p. equivalence relation on a standard probability space. Let $\theta \in [[\mathcal{R}]]$ and denote $A = \mathrm{dom}(\theta)$ and $B = \mathrm{im}(\theta)$. Prove that there is $\widetilde{\theta} \in [\mathcal{R}]$ such that $\widetilde{\theta}_{|A} = \theta$.
Suggestion: Prove that for almost every $x \in B \setminus A$, there exists $n := n(x) \in \mathbb{N}$ such that we have

$\theta^{-1}(x), ..., \theta^{-(n-1)} \in A \cap B$ and $\theta^{-n}(x) \in A \setminus B$. Define

$$\widetilde{\theta}(x) = \begin{cases} \theta(x), \text{ if } x \in A, \\ \theta^{-n(x)}(x), \text{ if } x \in B \setminus A, \text{ and} \\ x, \text{ if } x \in X \setminus (A \cup B). \end{cases}$$

Definition 1.18. Let $\mathcal{R}$ be a p.m.p. equivalence relation on a standard probabilty space (X, μ) and let $A \subset X$ be a non-null Borel set. Endow A with the σ-algebra of Borel subsets of X which are contained in A, and the probability measure $\mu_A(Y) = \mu(Y)/\mu(A)$, for every Borel set $Y \subset A$. Note that (A, μ_A) is a standard probability space (see [Ke95, Section 13]). The *restriction* of $\mathcal{R}$ to A is defined by $\mathcal{R}|A := \mathcal{R} \cap (A \times A)$.

Exercise 1.19. Prove that

(1) $\mathcal{R}|A$ is a countable p.m.p. equivalence relation on (A, μ_A).
(2) If $\mathcal{R}$ is ergodic, then $\mathcal{R}|A$ is ergodic.

1.5. **Orbit equivalence.** Two p.m.p. equivalence relations $\mathcal{R}$ and $\mathcal{S}$ on standard probability spaces (X, μ) and (Y, ν) are *isomorphic* to $\mathcal{S}$ if there is a m.p. isomorphism $\theta : X \to Y$ such that $\theta([x]_{\mathcal{R}}) = [\theta(x)]_{\mathcal{S}}$, for almost every $x \in X$.

Definition 1.20. Two p.m.p. actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ of countable groups Γ and Λ are *orbit equivalent* if their orbit equivalence relations are isomorphic. In other words, there is a m.p. isomorphism $\theta : X \to Y$ such that $\theta(\Gamma \cdot x) = \Lambda \cdot \theta(x)$, for almost every $x \in X$. The actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ are *conjugate* if there are a group isomorphism $\delta : \Gamma \to \Lambda$ and a m.p. isomorphism $\theta : X \to Y$ such that $\theta(g \cdot x) = \delta(g) \cdot \theta(x)$, for all $g \in \Gamma$ and almost every $x \in X$.

Orbit equivalence is a much coarser notion of equivalence for p.m.p. actions than conjugacy, as illustrated by the three exercises below and the results of the following two sections.

The first exercise gives examples of actions of non-isomorphic groups which are orbit equivalent.

Exercise 1.21. Let $(X, \mu) = (\{0,1\}^{\mathbb{N}}, \nu^{\otimes \mathbb{N}})$, where ν is the probability measure on $\{0,1\}$ given by $\nu(\{0\}) = \nu(\{1\}) = 1/2$. Let $\Gamma = \oplus_{n \in \mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ and the p.m.p. action $\Gamma \curvearrowright (X, \mu)$ given by $(g_n) \cdot (x_n) = (y_n)$, where $y_n \equiv g_n + x_n \pmod 2$. In other words, the n-th copy of $\mathbb{Z}/2\mathbb{Z}$ flips the n-th coordinate in X. Let $T : X \to X$ be the m.p. isomorphism (called an *odometer*) given by adding 1 to the left of an infinite sequence of 0's and 1's. If $x = (x_n) \in X$ has $x_1 = ... = x_k = 1$ and $x_{k+1} = 0$, then $T(x) = (y_n)$, where $y_1 = ... = y_k = 0$, $y_{k+1} = 1$ and $y_n = x_n$, for $n > k + 1$. Consider the p.m.p. action $\mathbb{Z} \curvearrowright (X, \mu)$, where the generator of $\mathbb{Z}$ acts through T: $n \cdot x = T^n(x)$.

Prove that the actions $\Gamma \curvearrowright (X, \mu)$ and $\mathbb{Z} \curvearrowright (X, \mu)$ have almost the same orbits.

The second exercise shows that conjugacy of left translation actions is very restrictive.

Exercise 1.22. Let Γ and Λ be countable dense subgroups of compact metrizable groups G and H. Denote by $\mu := \mathrm{m}_G$ and $\nu := \mathrm{m}_H$ the Haar measures of G and H. Assume $\delta : \Gamma \to \Lambda$ is a group isomorphism and $\theta : G \to H$ is a Borel map such that $\theta(gx) = \delta(g)\theta(x)$, for almost every $x \in G$.

(1) If $k \in G$, prove that there is $\pi(k) \in H$ such that $\theta(xk) = \theta(x)\pi(k)$, for almost every $x \in G$.
(2) Prove that $\pi : G \to H$ is a homomorphism.
(3) Prove that there exists $h \in H$ such that $\theta(x) = h\pi(x)$, for almost every $x \in G$.
(4) Prove that $\delta(g) = h\pi(g)h^{-1}$, for every $g \in \Gamma$.
(5) Prove that $\pi : G \to H$ is a continuous group isomorophism.
(6) Deduce that the left translation actions $\Gamma \curvearrowright (G, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ are conjugate if and only if there is a continuous group isomorphism $\delta : G \to H$ such that $\delta(\Gamma) = \Lambda$.

Suggestion for (1). Define $\theta_k : G \to H$ by letting $\theta_k(x) = \theta(x)^{-1}\theta(xk)$. Note that θ_k is invariant under the left translation action $\Gamma \curvearrowright (G, \mu)$, and apply Lemma 4.16 (which we will prove later)
Suggestion for (3). Note that $\theta(xk) = \theta(x)\pi(k)$, for almost every $(x, k) \in G \times G$, and apply Fubini's theorem.

The third exercise gives examples of non-conjugate actions of $\mathbb{Z}$. These actions are orbit equivalent as a consequence of Dye's theorem 2.1.

Exercise 1.23. Let $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ be the group of complex numbers of modulus 1 endowed with the topology induced from $\mathbb{C}$. Then $\mathbb{T}$ is a compact metrizable groups. Denote by λ its Haar measure (which coincides with the normalized Lebesgue measure). Let $\alpha \in \mathbb{R} \setminus \pi\mathbb{Q}$ be an irrational multiple of π. Then the map $n \mapsto \exp(in\alpha)$ gives a dense embedding of $\mathbb{Z}$ into $\mathbb{T}$. We denote by σ_α the associated left translation action $\mathbb{Z} \curvearrowright (\mathbb{T}, \lambda)$, called the *irrational rotation by angle* α, i.e.,

$$n \cdot z = \exp(in\alpha)z.$$

(1) Prove that if $\delta : \mathbb{T} \to \mathbb{T}$ is a continuous group isomorphism, then $\delta(z) = z$, for every $z \in \mathbb{T}$, or $\delta(z) = 1/z$, for every $z \in \mathbb{T}$.
(2) Let $\alpha, \beta \in \mathbb{R} \setminus \pi\mathbb{Q}$. Prove that σ_α is conjugate to σ_β if and only if $\exp(i\alpha) = \exp(i\beta)$ or $\exp(i\alpha) = 1/\exp(i\beta)$. (This proves a result due to Halmos and von Neumann [HvN42]).

2. Dye's theorem and hyperfiniteness

Theorem 2.1 (Dye, [Dy59])**.** *Any two ergodic p.m.p. actions of* $\mathbb{Z}$ *on a non-atomic standard probability space are orbit equivalent.*

Definition 2.2. A p.m.p. equivalence relation $\mathcal{R}$ on a standard probability space (X, μ) is called

(1) *finite* if $[x]_{\mathcal{R}}$ is finite, for almost every $x \in X$,
(2) of *type* I_n, for some $n \in \mathbb{N}$, if $[x]_{\mathcal{R}}$ has exactly n elements, for almost every $x \in X$, and
(3) *hyperfinite* if there are a co-null set $Y \subset X$ and an increasing sequence $\mathcal{R}_1 \subset \mathcal{R}_2 \subset ...$ of finite p.m.p. subequivalence relations of $\mathcal{R}|Y$ such that $\mathcal{R}|Y = \cup_{k\in\mathbb{N}}\mathcal{R}_k$.

Theorem 2.3 (Dye, [Dy59])**.** *Any two ergodic hyperfinite p.m.p. equivalence relations on a non-atomic standard probability space are orbit equivalent.*

Proof of Theorem 2.1 assuming Theorem 2.3. Let $\mathbb{Z} \curvearrowright (X, \mu)$ be an ergodic p.m.p. action. By Theorem 2.3, in order to get the conclusion, it suffices to argue that $\mathcal{R} := \mathcal{R}(\mathbb{Z} \curvearrowright X)$ is hyperfinite. This a consequence of the so-called Rokhlin's lemma (see [KM04, Section 7]). We provide a short, alternative argument. To this end, denote by $T : X \to X$ the m.p. isomorphism corresponding to a generator of $\mathbb{Z}$. Then T is ergodic and $\mathcal{R} = \{(x, T^n x) \mid x \in X, n \in \mathbb{Z}\}$.

Let $X_1 \subset X_2 \subset ...$ be an increasing sequence of Borel subsets of X with $\mu(X_k) = 1 - 1/k$. We claim that the set $Y_k = \{x \in X_k \mid T^n x \in X_k, \text{ for every } n \geq 0\}$ is null. Since $T(Y_k) \subset Y_k$, $\mu(T(Y_k)) = \mu(Y_k) < 1$ and T is ergodic, Exercise 1.13 implies that Y_k is null. Similarly, the set $Z_k = \{x \in X_k \mid T^n x \in X_k, \text{ for every } n \leq 0\}$ is null.

Let $\mathcal{S}_k$ be the equivalence relation on X_k defined as follows: $(x, y) \in \mathcal{S}_k$ iff there is $n \geq 0$ such that $T^n x = y$ and $x, Tx, T^2x, ..., T^n x \in X_k$ or $n \leq 0$ such that $T^n x = y$ and $x, T^{-1}x, T^{-2}x, ..., T^n x \in X_k$. Then $\mathcal{S}_k \subset \mathcal{S}_{k+1}$ and the restriction of $\mathcal{S}_k$ to $X_k \setminus (Y_k \cup Z_k)$ is a finite equivalence relation. Thus, if

$$\mathcal{R}_k = \mathcal{S}_k \cup \{(x, x) \mid x \in X \setminus X_k\},$$

then $\mathcal{R}_k \subset \mathcal{R}_{k+1} \subset \mathcal{R}$, and the restriction of $\mathcal{R}_k$ to $X \setminus (Y_k \cup Z_k)$ is a finite equivalence relation.

Finally, note that if $k, m \in \mathbb{N}$, then $\mu(\cap_{i=-m}^{m} T^{-i}X_k) \geq 1 - (2m+1)/k$. Thus, the set

$$W = \bigcap_{m\in\mathbb{N}} (\bigcup_{k\in\mathbb{N}} (\bigcap_{i=-m}^{m} T^{-i}X_k))$$

is co-null in X. Moreover, $\mathcal{R}|W = \cup_{k\in\mathbb{N}}\mathcal{R}_k|W$. Thus, $V = W \setminus (\cup_{k\in\mathbb{N}}(Y_k \cup Z_k))$ is co-null in X, $\mathcal{R}_k|V$ is a finite equivalence relation, and $\mathcal{R}|V = \cup_{k\in\mathbb{N}}\mathcal{R}_k|V$. This proves that $\mathcal{R}$ is hyperfinite. ■

In preparation for the proof of Theorem 2.3, we introduce the following definitions:

Definition 2.4. A *fundamental domain* for a p.m.p equivalence relation $\mathcal{R}$ on (X,μ) is a Borel set $Y \subset X$ that intersects almost every $\mathcal{R}$-class in exactly one point. An *array* for $\mathcal{R}$ is a set $\{X_1, \varphi_1, ...\varphi_n\}$ consisting of a Borel set $X_1 \subset X$ of measure $1/n$ and partial m.p. isomorphisms $\varphi_i : X_1 \to \varphi_i(X_1)$ such that $\varphi_1 = \mathrm{Id}_{X_1}$, $\{\varphi_i(X_1)\}_{i=1}^n$ is a partition of X, and almost every $\mathcal{R}$-class is equal to $\{\varphi_i(x)\}_{i=1}^n$, for some $x \in X_1$ (see [KM04] and the references therein).

Exercise 2.5. Prove that the following hold for an p.m.p. equivalence relation $\mathcal{R}$ on (X,μ):

(1) $\mathcal{R}$ is finite iff it admits a fundamental domain.
(2) $\mathcal{R}$ is of type I_n iff it admits an array $\{X_1, \varphi_1, ..., \varphi_n\}$.
(3) $\mathcal{R}$ is of type I_n iff it is the orbit equivalence relation of a free p.m.p. action $\mathbb{Z}/n\mathbb{Z} \curvearrowright (X,\mu)$.

Exercise 2.6. Let $\mathcal{R}$ and $\mathcal{S}$ be p.m.p. equivalence relations of types I_n and I_m on (X,μ) such that $\mathcal{R} \subset \mathcal{S}$. Let $X_1 \subset X$ be a fundamental domain for $\mathcal{R}$. Prove that $n \mid m$ and $\mathcal{S}|X_1$ is of type $\mathrm{I}_{m/n}$.

Assumption. For the rest of this section, $\mathcal{R}$ is an ergodic p.m.p. equivalence relation on (X,μ).

Definition 2.7. Let $\mathcal{S}, \mathcal{T}$ be subequivalence relations of $\mathcal{R}$. We write $\mathcal{S} \subset_\varepsilon \mathcal{T}$ (and say that $\mathcal{S}$ is *ε-contained* in $\mathcal{T}$) for $\varepsilon > 0$ if there is a Borel set $Y \subset X$ such that $\mathcal{S}|Y \subset \mathcal{T}|Y$ and $\mu(X \setminus Y) < \varepsilon$.

The proof of Theorem 2.3 relies on several lemmas.

Lemma 2.8. *Let $\mathcal{R}_1 \subset \mathcal{R}_2 \subset ...$ be a sequence of subequivalence relations of $\mathcal{R}$ with $\mathcal{R} = \cup_{k\in\mathbb{N}}\mathcal{R}_k$. Then for every finite subequivalence relation $\mathcal{S} \subset \mathcal{R}$ and $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that $\mathcal{S} \subset_\varepsilon \mathcal{R}_k$.*

Exercise 2.9. Prove Lemma 2.8.

Lemma 2.10. *Let $\mathcal{S} \subset \mathcal{R}$ be a finite subequivalence relation and $\delta > 0$. Then there is a subequivalence relation $\mathcal{T} \subset \mathcal{R}$ of type I_{2^k}, for some $k \in \mathbb{N}$, such that $\mathcal{S} \subset_\delta \mathcal{T}$.*

Proof. We prove the lemma when $\mathcal{S}$ is of type I_d, for some $d \in \mathbb{N}$, and leave the general case as an exercise. In this case, by Exercise 2.5, $\mathcal{S}$ admits an array $\{X_1, \varphi_1, ..., \varphi_d\}$.

Let $Y_1 \subset X_1$ be a Borel set such that $\mu(Y_1) = m/2^k$, for some $m, k \in \mathbb{N}$, and $\mu(X_1 \setminus Y_1) < \delta/d$. Partition $Y_1 = \sqcup_{j=1}^m Z_j$ into Borel sets of measure $1/2^k$. Let $Y = \sqcup_{i=1}^d \varphi_i(Y_1)$. Then $\mu(Y) = dm/2^k$, hence $\mu(X \setminus Y) = (2^k - dm)/2^k$. Partition $X \setminus Y = \sqcup_{l=1}^{2^k - dm} W_l$ into Borel sets of measure $1/2^k$.

Let $\psi_1 = \mathrm{Id}_{Z_1}$. By Lemma 1.16, for every $2 \leq j \leq m$ and $1 \leq l \leq 2^n - dm$, we can find $\psi_j, \rho_l \in [[\mathcal{R}]]$ such that $\mathrm{dom}(\psi_j) = \mathrm{dom}(\rho_l) = Z_1$, $\mathrm{im}(\psi_j) = Z_j$ and $\mathrm{im}(\rho_l) = W_l$. Let $\mathcal{T} \subset \mathcal{R}$ be the equivalence relation whose classes are the sets $\{\varphi_i(\psi_j(x)) \mid 1 \leq i \leq d, 1 \leq j \leq m\} \cup \{\rho_l(x) \mid 1 \leq l \leq 2^k - dm\}$ with $x \in Z_1$. Then $\mathcal{T}$ is of type I_{2^k} and $\mathcal{S}|Y \subset \mathcal{T}|Y$. Since $\mu(X \setminus Y) < \delta$, this proves the lemma. ■

Lemma 2.11. *Let $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ be finite subequivalence relations of $\mathcal{R}$. Assume that $\mathcal{S}$ is of type I_n and $\mathcal{S} \subset_\varepsilon \widetilde{\mathcal{S}}$, for some $n \in \mathbb{N}$ and $\varepsilon > 0$. Then there is a subequivalence relation $\mathcal{T} \subset \mathcal{R}$ such that*

(1) $\mathcal{T}$ is of type $\mathrm{I}_{n\cdot 2^k}$, for some $k \in \mathbb{N}$,
(2) $\mathcal{S} \subset \mathcal{T}$, and

(3) $\widetilde{\mathcal{S}} \subset_{n\cdot\varepsilon} \mathcal{T}$.

Proof. Let $Y \subset X$ be a Borel set such that $\mathcal{S}|Y \subset \widetilde{\mathcal{S}}|Y$ and $\mu(X \setminus Y) < \varepsilon$. Since $\mathcal{S}$ is of type I_n, we can write $\mathcal{S} = \mathcal{R}(\mathbb{Z}/n\mathbb{Z} \curvearrowright X)$ by Exercise 2.5. Then $Z = \cap_{g\in\mathbb{Z}/n\mathbb{Z}}(g \cdot Y)$ is $\mathcal{S}$-invariant and $\mu(X \setminus Z) < n \cdot \varepsilon$. Thus, $\mathcal{S}|Z$ is of type I_n, hence by Exercise 2.5 it admits an array $\{Z_1, \varphi_1, ..., \varphi_n\}$.

Let $\delta > 0$ such that $\mu(X \setminus Z) + n \cdot \delta < n \cdot \varepsilon$. Since $\widetilde{\mathcal{S}}|Z_1 \subset \mathcal{R}|Z_1$ is a finite subequivalence relation and $\mathcal{R}|Z_1$ is ergodic, by Lemma 2.10 there is a subequivalence relation $\mathcal{T}_0 \subset \mathcal{R}|Z_1$ of type I_{2^k}, for some $k \in \mathbb{N}$, and a Borel set $W_1 \subset Z_1$ such that $\widetilde{\mathcal{S}}|W_1 \subset \mathcal{T}_0|W_1$ and $\mu(Z_1 \setminus W_1) < \delta$.

Let $\mathcal{T}_1 \subset \mathcal{R}|Z$ be the subequivalence relation whose classes are the sets $\sqcup_{i=1}^n \varphi_i([x]_{\mathcal{T}_0})$ with $x \in Z_1$. Then $\mathcal{T}_1$ is a equivalence relation of type $\mathrm{I}_{n\cdot 2^k}$ which contains $\mathcal{S}|Z$. Since $\widetilde{\mathcal{S}}|W_1 \subset \mathcal{T}_0|W_1$, we also get that $\widetilde{\mathcal{S}}|W \subset \mathcal{T}_1|W$, where $W := \sqcup_{i=1}^n \varphi_i(W_1) \subset Z$.

Finally, since Z is $\mathcal{S}$-invariant, we have that $\mathcal{S}|(X \setminus Z)$ is of type I_n. Since $\mathcal{R}$ is ergodic, we can find a subequivalence relation $\mathcal{T}_2 \subset \mathcal{R}|(X \setminus Z)$ of type $\mathrm{I}_{n\cdot 2^k}$ which contains $\mathcal{S}|(X \setminus Z)$. Then $\mathcal{T} := \mathcal{T}_1 \sqcup \mathcal{T}_2$ is a subequivalence of $\mathcal{R}$ of type $\mathrm{I}_{n\cdot 2^k}$ which contains $\mathcal{S}$. Moreover, $\widetilde{\mathcal{S}}|W \subset \mathcal{T}|W$. Since $\mu(Z \setminus W) = n \cdot \mu(Z_1 \setminus W_1) < n \cdot \delta$, we get that

$$\mu(X \setminus W) = \mu(X \setminus Z) + \mu(Z \setminus W) < \mu(X \setminus Z) + n \cdot \delta < n \cdot \varepsilon$$

and the conclusion follows. ■

Next, we show that the equivalence relation $\mathcal{T}$ from Lemma 2.11 can be chosen to have a certain additional property. Let $\mathcal{S} \subset \mathcal{T}$ be p.m.p. equivalence relations of type I_n and I_m on (X, μ). Let $\mathcal{A} = \{X_1, \varphi_i\}$ be an array for $\mathcal{S}$. A Borel set $Z \subset X$ is *ε-contained* in $\mathcal{A}$ if $\mu(Z\triangle(\cup_{i\in F}\varphi_i(X_1)) < \varepsilon$, for some $F \subset \{1, .., n\}$. An array $\mathcal{A}'$ for $\mathcal{T}$ *refines* $\mathcal{A}$ if there is an array $\mathcal{B} = \{Y_1, \psi_j\}$ for $\mathcal{T}|X_1$ such that $\mathcal{A}' = \{Y_1, \varphi_i \circ \psi_j\}$. We write $\mathcal{A}' = \mathcal{A} \vee \mathcal{B}$.

Lemma 2.12. *Let $\mathcal{T} \subset \mathcal{R}$ be a subequivalence relation of type I_n with an array $\mathcal{A} = \{X_1, \varphi_1, ..., \varphi_n\}$. Let $Z \subset X$ be a Borel set and $\delta > 0$. Then there are an equivalence relation $\mathcal{T} \subset \widetilde{\mathcal{T}} \subset \mathcal{R}$ of type $\mathrm{I}_{n\cdot 2^m}$, for $m \geq 0$, and an array $\mathcal{B} = \{Y_1, \psi_1, ..., \psi_{n\cdot 2^m}\}$ for $\widetilde{\mathcal{T}}$ which refines $\mathcal{A}$ and δ-contains Z.*

Proof. For every large enough m, we can find a partition $X_1 = \sqcup_{l=1}^{2^m} X_{1,l}$ into Borel sets of measure $\mu(X_1)/2^m = 1/(n \cdot 2^m)$ and sets $S_i \subset \{1, 2, ..., 2^m\}$ such that $\mu(\varphi_i^{-1}(Z \cap X_i)\triangle(\cup_{l\in S_i} X_{1,l})) < \delta/n$, for every $1 \leq i \leq n$. Thus, we have

$$\mu\big((Z \cap X_i)\triangle(\cup_{l\in S_i}\varphi_i(X_{1,l}))\big) < \delta/n, \text{ for every } 1 \leq i \leq n. \tag{2.1}$$

Since $\mathcal{R}|X_1$ is ergodic, Lemma 1.16 gives a subequivalence $\mathcal{T}' \subset \mathcal{R}|X_1$ of type I_{2^m} with an array $\{X_{1,1}, \psi_1, ..., \psi_{2^m}\}$ satisfying $\psi_l(X_{1,1}) = X_{1,l}$, for all $1 \leq l \leq 2^m$. Let $\widetilde{\mathcal{T}}$ be the type $\mathrm{I}_{n\cdot 2^m}$ equivalence relation with $\mathcal{B} = \{X_{1,1}, \varphi_i \circ \psi_l\}$ as an array. Then $\mathcal{T} \subset \widetilde{\mathcal{T}}$, $\mathcal{B}$ refines $\mathcal{A}$, and (2.1) implies that

$$\mu\big(Z\triangle(\cup_{1\leq i\leq n, l\in S_i}\varphi_i(X_{1,l}))\big) < \delta,$$

which finishes the proof. ■

Proof of Theorem 2.3. After replacing $\mathcal{R}$ with a co-null subset, we can find an increasing sequence $\mathcal{R}_1 \subset \mathcal{R}_2 \subset ...$ of finite subequivalence relations of $\mathcal{R}$ such that $\mathcal{R} = \cup_{k\in\mathbb{N}}\mathcal{R}_k$. Since X is a standard Borel space, we can find sequence of Borel subsets $\{Y_n\}_{n\in\mathbb{N}}$ which separate points in X, and in which every Y_n repeats infinitely many times.

Claim 2.13. There is an increasing sequence $\mathcal{S}_1 \subset \mathcal{S}_2 \subset ...$ of subequivalence relations of $\mathcal{R}$ and arrays $\mathcal{A}_n$ for $\mathcal{S}_n$ such that for all $n \in \mathbb{N}$

(1) $\mathcal{R}_n \subset_{1/2^n} \mathcal{S}_n$,

(2) $\mathcal{S}_n$ is of type $\mathrm{I}_{2^{d_n}}$, for some $d_n \in \mathbb{N}$,
(3) $\mathcal{A}_n$ refines $\mathcal{A}_{n-1}$ (where $\mathcal{A}_0 = \{X, \mathrm{Id}_X\}$), and
(4) Y_n is $1/2^n$-contained in $\mathcal{A}_n$.

Proof of Claim 2.13. We proceed by induction on n.

By Lemma 2.10 there is an equivalence relation $\mathcal{T}_1$ of type $\mathrm{I}_{2^{k_1}}$ such that $\mathcal{R}_1 \subset_{1/2} \mathcal{T}_1$. By Lemma 2.12 we can find an equivalence relation $\mathcal{T}_1 \subset \mathcal{S}_1$ of type $\mathrm{I}_{2^{d_1}}$, for $d_1 \geq k_1$, and an array $\mathcal{A}_1$ for $\mathcal{S}_1$ which $1/2$-contains Y_1. This proves the case $n = 1$.

Next, given $\mathcal{S}_1, ..., \mathcal{S}_{n-1}, \mathcal{A}_1, ..., \mathcal{A}_{n-1}$, for some $n \geq 2$, let us construct $\mathcal{S}_n, \mathcal{A}_n$. First, by applying Lemma 2.8 we can find $k_n \geq n$ such that $\mathcal{S}_{n-1} \subset_{1/2^{n+d_{n-1}}} \mathcal{R}_{k_n}$. Second, since $\mathcal{S}_{n-1}$ is of type $\mathrm{I}_{2^{d_{n-1}}}$, by applying Lemma 2.11 (with $\varepsilon = 1/2^{n+d_{n-1}}$), we can find a subequivalence relation $\mathcal{T}_n \subset \mathcal{R}$ of type $\mathrm{I}_{2^{k_n}}$, for some $k_n \geq d_{n-1}$, such that $\mathcal{S}_{n-1} \subset \mathcal{T}_n$ and $\mathcal{R}_{k_n} \subset_{1/2^n} \mathcal{T}_n$. Let $\mathcal{B}_n$ be an array of $\mathcal{T}_n$ which refines $\mathcal{A}_{n-1}$. Third, Lemma 2.12 gives a subequivalence relation $\mathcal{S}_n \subset \mathcal{R}$ of type $\mathrm{I}_{2^{d_n}}$, for some $d_n \geq k_n$, such that $\mathcal{T}_n \subset \mathcal{S}_n$ and an array $\mathcal{A}_n$ for $\mathcal{T}_n$ which refines $\mathcal{B}_n$ and $1/2^n$-contains Y_n. Then $\mathcal{A}_n$ also refines $\mathcal{A}_{n-1}$. As $k_n \geq n$, we have $\mathcal{R}_n \subset \mathcal{R}_{k_n} \subset_{1/2^n} \mathcal{T}_n \subset \mathcal{S}_n$, and thus condition (1) holds. This proves the inductive step. □

Next, since $\mathcal{R} = \cup_{n\in\mathbb{N}}\mathcal{R}_n$, condition (1) from Claim (1) implies that $\mathcal{R}|Y = \cup_{n\in\mathbb{N}}\mathcal{S}_n|Y$, for a co-null subset $Y \subset X$. Thus, after replacing X with Y, we may assume that $\mathcal{R} = \cup_{n\in\mathbb{N}}\mathcal{S}_n$.

In the rest of the proof, we follow the exposition from the proof of [KM04, Theorem 7.13]. Write $\mathcal{A}_1 = \{X_1, \varphi_{1,1}, ..., \varphi_{1,2^{d_1}}\}$. Since $\mathcal{A}_n$ refines $\mathcal{A}_{n-1}$, we can find an array $\mathcal{B}_n = \{X_n, \varphi_{n,1}, ..., \varphi_{n,2^{d_n-d_{n-1}}}\}$ for $\mathcal{S}_n|X_{n-1}$ such that $\mathcal{A}_n = \mathcal{A}_{n-1} \vee \mathcal{B}_n$, for every $n \geq 2$. Put $d_0 = 0$. Then we have

$$\mathcal{A}_n = \{X_n, \varphi_{1,k_1} \circ \varphi_{2,k_2} \circ ... \circ \varphi_{n,k_n}, 1 \leq k_i \leq 2^{d_i - d_{i-1}}, 1 \leq i \leq n\}.$$

We let $X(k_1, ..., k_n) = (\varphi_{1,k_1} \circ \varphi_{2,k_2} \circ ... \circ \varphi_{n,k_n})(X_n)$, for every $n \in \mathbb{N}$ and $(k_1, ..., k_n) \in \prod_{1\leq i\leq n}\{1, ..., 2^{d_i-d_{i-1}}\}$. Then we have

$$X(k_1, ..., k_n, k_{n+1}) \subset X(k_1, ..., k_n) \quad \text{and} \quad X = \sqcup_{k_1,...,k_n} X(k_1, ..., k_n). \tag{2.2}$$

For $m \in \mathbb{N}$, let μ_m be the uniform probability measure on $\{1, ..., m\}$. Define a p.m.p equivalence relation $\mathcal{S}$ on

$$(Y, \nu) = \prod_{n\in\mathbb{N}} \left(\{1, ..., 2^{d_n-d_{n-1}}\}, \mu_{2^{d_n-d_{n-1}}}\right)$$

by

$$(k_n)\mathcal{S}(l_n) \iff \text{there is } N \in \mathbb{N} \text{ such that } k_n = l_n, \text{ for all } n \geq N.$$

By (2.2), if $x \in X$, then there is a sequence $(k_n) \in Y$ such that $x \in X(k_1, ..., k_n)$, for every $n \in \mathbb{N}$. We define $\theta : X \to Y$ by letting $\theta(x) = (k_n)$.

Claim 2.14. θ is Borel, measure preserving and 1-1 on a co-null $\mathcal{R}$-invariant Borel set $X_0 \subset X$.

Proof of Claim 2.14. For $n \in \mathbb{N}$ and $(k_1, ..., k_n) \in \prod_{1\leq i\leq n}\{1, ..., 2^{d_i-d_{i-1}}\}$, consider the cylinder set

$$C(k_1, ..., k_n) = \{(l_i) \in Y \mid l_i = k_i, \text{ for all } i \leq n\}.$$

Then $\theta^{-1}(C(k_1, ..., k_n)) = X(k_1, ..., k_n)$. Since X_n is a fundamental domain for $\mathcal{S}_n$, we have

$$\mu(X(k_1, ..., k_n)) = \mu(X_n) = 1/2^{d_n} = \nu(C(k_1, ..., k_n)).$$

This implies that θ is Borel and measure preserving.

Now, by condition (4) from Claim 2.13, for every $n \in \mathbb{N}$, we can find a set $Y_n' \subset X$ which is a finite union of sets of the form $X(k_1, ..., k_n)$ such that $\mu(Y_n' \triangle Y_n) < 1/2^n$. Then

$$N = \bigcap_{m \in \mathbb{N}} \bigcup_{n > m} (Y_n' \triangle Y_n)$$

is a null Borel set and the restriction of θ to $X \setminus N$ is 1-1. To see this, let $x \neq x' \in X \setminus N$. Then there is $m \in \mathbb{N}$, such that $x, x' \notin (Y_n' \triangle Y_n)$, for any $n > m$. Let $n > m$ such that Y_n separates x and x'. Then it follows that Y_n' separates x and x', which implies that $\theta(x) \neq \theta(x')$.

By Theorem 1.10, we can write $\mathcal{R} = \mathcal{R}(\Gamma \curvearrowright X)$, for a countable group Γ. Then $X_0 := \cap_{g \in \Gamma} g \cdot (X \setminus N)$ is a co-null, $\mathcal{R}$-invariant subset of X on which θ is 1-1. □

Claim 2.15. For all $x, x' \in X_0$ we have that $(x, x') \in \mathcal{R} \iff (\theta(x), \theta(x')) \in \mathcal{S}$.

Proof of Claim 2.15. Let $x, x' \in X$ such that $(x, x') \in \mathcal{R}$, and write $\theta(x) = (k_n)$ and $\theta(x) = (k_n')$. Then there is $n \in \mathbb{N}$ such that $(x, x') \in \mathcal{S}_n$. Since X_n is a fundamental domain for $\mathcal{S}_n$, we can find $y \in X_n$ such that $x = (\varphi_{1,k_1} \circ ... \circ \varphi_{n,k_n})(y)$ and $x' = (\varphi_{1,k_1'} \circ ... \circ \varphi_{n,k_n'})(y)$. Consider the unique element $k \in \{1, ..., 2^{d_{n+1} - d_n}\}$ such that $y \in \varphi_{n+1,k}(X_{n+1})$. Then $k_{n+1} = k_{n+1}' = k$. By induction on m, we derive that $k_m = k_m'$, for all $m > n$, which implies that $(\theta(x), \theta(x')) \in \mathcal{S}$.

Let $x, x' \in X_0$ such that $(\theta(x), \theta(x')) \in \mathcal{S}$. Thus, if $\theta(x) = (k_n)$ and $\theta(x') = (k_n')$, there is $m \in \mathbb{N}$ such that $k_n = k_n'$, for all $n > m$. Let $y \in X_m$ such that $x = (\varphi_{1,k_1} \circ .... \circ \varphi_{m,k_m})(y)$, and put $x'' = (\varphi_{1,k_1'} \circ .... \circ \varphi_{m,k_m'})(y)$. Then $\theta(x'') = (k_n') = \theta(x')$. Since $(x, x'') \in \mathcal{R}$ and X_0 is $\mathcal{R}$-invariant, we get that $x'' \in X_0$. Since $x' \in X_0$ and θ is 1-1 on X_0, we get that $x' = x''$, hence $(x, x') \in \mathcal{R}$. □

Finally, Claims 2.14 and 2.15 imply that $\mathcal{R}$ is isomorphic to $\mathcal{S}$. Since $\mathcal{S}$ is isomorphic to the equivalence relation on $(\{1,2\}^{\mathbb{N}}, \mu_2^{\otimes \mathbb{N}})$ defined by the same formula as $\mathcal{S}$, the conclusion follows. ■

3. Amenability

3.1. **Amenable groups.** A countable group Γ is called *amenable* if there exists a linear functional $\varphi : \ell^\infty(\Gamma) \to \mathbb{C}$ which is unital ($\varphi(\mathbf{1}_X) = 1$), positive ($\varphi(f) \geq 0$, for every $f \in \ell^\infty(\Gamma)$, $f \geq 0$) and left translation invariant: $\varphi(g \cdot f) = \varphi(f)$, for all $g \in \Gamma$ and $f \in \ell^\infty(\Gamma)$, where $(g \cdot f)(h) = f(g^{-1}h)$.

Exercise 3.1. Let I be a set and $\varphi : \ell^\infty(I) \to \mathbb{C}$ be a unital positive linear functional. Prove that $|\varphi(f)| \leq \|f\|_\infty$, for every $f \in \ell^\infty(I)$.

Exercise 3.2. Let Γ be a countable group and denote by $\mathcal{P}(\Gamma)$ the collection of all subsets of Γ. Prove that Γ is amenable if and only if there exists a finitely additive measure $m : \mathcal{P}(\Gamma) \to [0,1]$ such that $m(\Gamma) = 1$ and $m(gA) = m(A)$, for every $g \in \Gamma$ and $A \subset \Gamma$, where $gA := \{gx \mid x \in A\}$.

Example 3.3. Every finite group Γ is amenable, as witnessed by the functional $\varphi(f) = |\Gamma|^{-1} \sum_{g \in \Gamma} f(g)$.

In order to give examples of infinite amenable groups, we need to recall the following:

Definition 3.4. A *free ultrafilter* on $\mathbb{N}$ is a unital homomorphism $\omega : \ell^\infty(\mathbb{N}) \to \mathbb{C}$ which is not the evaluation e_n at any $n \in \mathbb{N}$ given by $e_n(f) = f(n)$.

Remark 3.5. To see that free ultrafilters on $\mathbb{N}$ exist, let $K_n \subset \ell^\infty(\mathbb{N})^*$ be the weak* closure of $\{e_k \mid k > n\}$. Then K_n is weak*-compact by Alaoglu's theorem (see [Fo99, Theorem 5.18]) and $K_{n+1} \subset K_n$, for all n. Thus, $\cap_n K_n \neq \emptyset$. If $\omega \in \cap_n K_n$, then $\omega : \ell^\infty(\mathbb{N}) \to \mathbb{C}$ is a unital homomorphism. Moreover, if $n \in \mathbb{N}$, then $\omega \in K_n$, thus $\omega(\delta_n) = 0$ and hence $\omega \neq e_n$.

Notation. For a free ultrafilter ω on $\mathbb{N}$, we denote $\lim_{n \to \omega} x_n := \omega((x_n)_n)$, for every $(x_n)_n \in \ell^\infty(\mathbb{N})$.

Exercise 3.6. Prove that if ω is a free ultrafilter on $\mathbb{N}$ and $\lim_{n\to\infty} x_n = 0$, then $\lim_{n\to\omega} x_n = 0$.

Examples 3.7. (of amenable groups) Let ω be a free ultrafilter on $\mathbb{N}$.

(1) If $\Gamma = \cup_n \Gamma_n$, where $\Gamma_n < \Gamma$ are amenable subgroups and $\Gamma_n \subset \Gamma_{n+1}$, for all n, then Γ is amenable. Let $\varphi_n : \ell^\infty(\Gamma_n) \to \mathbb{C}$ be a unital, positive, left invariant linear functional. Define $\varphi : \ell^\infty(\Gamma) \to \mathbb{C}$ by $\varphi(f) = \lim_{n\to\omega} \varphi_n(f_{|\Gamma_n})$. Then φ is a left invariant state. Indeed, if $g \in \Gamma$, then $g \in \Gamma_N$, for some N. Thus, if $f \in \ell^\infty(\Gamma)$, then $(g\cdot f)_{|\Gamma_n} = g\cdot(f_{|\Gamma_n})$, for all $n \geq N$. Hence by Exercise 3.6 we get $\varphi(g \cdot f) = \lim_{n\to\omega} \varphi_n(g \cdot (f_{\Gamma_n})) = \lim_{n\to\omega} \varphi_n(f_{|\Gamma_n}) = \varphi(f)$. In particular, any increasing union of finite groups is amenable.

(2) $\mathbb{Z}$ is amenable. To see this, let $F_n = \{-n, -n+1, ..., -1, 0, 1, ..., n-1, n\}$. Then for any $g \in \mathbb{Z}$ we have $|(g + F_n) \setminus F_n| \leq 2|g|$ and thus $\lim_{n\to\infty} |(g + F_n) \setminus F_n|/|F_n| = 0$. Define $\varphi : \ell^\infty(\mathbb{Z}) \to \mathbb{C}$ by letting $\varphi(f) = \lim_{n\to\omega}(1/|F_n|)\sum_{x\in F_n} f(x)$. For every $g \in \mathbb{Z}$, we have

$$
\begin{aligned}
|\varphi(g \cdot f) - \varphi(f)| &= |\lim_{n\to\omega}(1/|F_n|)(\sum_{x\in F_n} f(x-g) - \sum_{x\in F_n} f(x))| \\
&\leq \|f\|_\infty \lim_{n\to\omega} |(F_n - g)\triangle F_n|/|F_n| = 0.
\end{aligned}
$$

Exercise 3.8. Prove that if Γ and Λ are countable amenable groups, then the direct product $\Gamma \times \Lambda$ is an amenable group. Prove that any countable abelian group is amenable.

Theorem 3.9. *Let Γ be a countable group. Then the following conditions are equivalent:*

(1) Γ is amenable.
(2) Γ satisfies the Reiter condition*: there exists a sequence of non-negative functions $f_n \in \ell^1(\Gamma)$ such that $\|f_n\|_1 = 1$, for all n, and $\lim_{n\to\infty} \|g \cdot f_n - f_n\|_1 = 0$, for all $g \in \Gamma$.*
(3) Γ satisfies the Følner condition*: there exists a sequence of finite subsets $F_n \subset \Gamma$ such that $\lim_{n\to\infty} |gF_n \setminus F_n|/|F_n| = 0$, for all $g \in \Gamma$.*

The proof of this result relies on two very useful tricks, due to Day (the proof of (1) $\Rightarrow$ (2)) and Namioka (the proof of (2) $\Rightarrow$ (3)). Namioka's trick uses the following useful identity:

Exercise 3.10. Let (S, ν) be a standard measure space (e.g., let (S, ν) a standard probability space or a countable set with its counting measure). Prove that if $f_1, f_2 \in L^1(S, \nu)$ and $f_1, f_2 \geq 0$, then

$$\|f_1 - f_2\|_1 = \int_0^\infty \|1_{\{f_1>t\}} - 1_{\{f_2>t\}}\|_1 \mathrm{d}t.$$

Proof of Theorem 3.9. Enumerate $\Gamma = \{g_n\}_{n\geq 1}$.

(1) $\Rightarrow$ (2) Fix $n \geq 1$ and consider the convex subset

$$C := \{(g_1 \cdot f - f, g_2 \cdot f - f, ..., g_n \cdot f - f) \mid f \in \ell^1(\Gamma), f \geq 0, \|f\|_1 = 1\}$$

of the Banach space $\ell^1(\Gamma)^{\oplus n}$ with the norm $\|(f_1, f_2, ..., f_n)\| = \sum_{i=1}^n \|f_i\|_1$.

We claim that $\mathbf{0} = (0, 0, ..., 0) \in \overline{C}^{\|\cdot\|}$. Assuming this claim, we can find $f_n \in \ell^1(\Gamma)$ such that $f_n \geq 0, \|f_n\|_1 = 1$ and $\sum_{i=1}^n \|g_i \cdot f_n - f_n\|_1 \leq 1/n$. This clearly implies (2).

If the claim were false, then since $\overline{C}^{\|\cdot\|} \subset \ell^1(\Gamma)^{\oplus n}$ is a closed convex set and $(\ell^1(\Gamma)^{\oplus n})^* = \ell^\infty(\Gamma)^{\oplus n}$, the Hahn-Banach separation theorem (see, e.g., [Ru91, Theorem 3.4]) implies the existence of $F_1, F_2, ..., F_n \in \ell^\infty(\Gamma)$ and $\alpha > 0$ such that $\sum_{i=1}^n \Re\langle g_i \cdot f - f, F_i\rangle \geq \alpha$, for any $f \in \ell^1(\Gamma)$ with $f \geq 0$ and $\|f\|_1 = 1$.

If we put $F = \sum_{i=1}^n \Re(g_i^{-1} \cdot F_i - F_i)$, then the last inequality rewrites as $\langle f, F\rangle \geq \alpha$, for any $f \in \ell^1(\Gamma)$ with $f \geq 0$ and $\|f\|_1 = 1$. For $f = \delta_g$, this implies that $F(g) \geq \alpha$, for all $g \in \Gamma$. Thus, we get that

$\varphi(F) \geq \varphi(\alpha \cdot 1) = \alpha > 0$. On the other hand, $\varphi(F) = \sum_{i=1}^n (\varphi(\Re(g_i^{-1} \cdot F_i)) - \varphi(\Re F_i)) = 0$. This gives the desired contradiction.

(2) $\Rightarrow$ (3) If $f_1, f_2 \in \ell^1(\Gamma)$ and $f_1, f_2 \geq 0$, then Exercise 3.10 gives that

$$\|f_1 - f_2\|_1 = \int_0^\infty \|1_{\{f_1 > t\}} - 1_{\{f_2 > t\}}\|_1 \mathrm{d}t \quad \text{and} \quad \|f_1\|_1 = \int_0^\infty \|1_{\{f_1 > t\}}\mathrm{d}t. \tag{3.1}$$

By (2), for any $n \geq 1$ we can find $f \in \ell^1(\Gamma)$ such that $f \geq 0, \|f\|_1 = 1$ and $\sum_{i=1}^n \|g_i \cdot f - f\|_1 < 1/n$. For $t > 0$, let $K_t = \{f > t\}$. Since $f \in \ell^1(\Gamma)$, we get that K_t is a finite subset of Γ. Also, note that $\{g \cdot f > t\} = gK_t$ and thus that $\|1_{\{g \cdot f > t\}} - 1_{\{f > t\}}\|_1 = |gK_t \triangle K_t|$, for all $g \in \Gamma$. Thus, by combining the last inequality with (3.1), we derive that

$$\int_0^\infty \sum_{i=1}^n |g_i K_t - K_t| \mathrm{d}t < 1/n = 1/n\|f\|_1 = \int_0^\infty (|K_t|/n)\mathrm{d}t.$$

Hence, there is $t_n > 0$ such that $F_n := K_{t_n}$ satisfies $\sum_{i=1}^n |g_i F_n \triangle F_n| < |F_n|/n$. This proves (3).

(3) $\Rightarrow$ (1) Let ω be a free ultrafilter on $\mathbb{N}$. Define $\varphi : \ell^\infty(\Gamma) \to \mathbb{C}$ by letting

$$\varphi(f) = \lim_{n \to \omega} \frac{1}{|F_n|} \sum_{x \in F_n} f(x).$$

Then as in the proof of Examples 3.7 (2), it follows that Γ is amenable. ∎

Proposition 3.11. $\mathbb{F}_2$ *is not amenable.*

Proof. Assume by contradiction that there exists a unital, positive, left translation invariant linear functional $\varphi : \ell^\infty(\mathbb{F}_2) \to \mathbb{C}$. Define $m : \mathcal{P}(\mathbb{F}_2) \to [0, 1]$ my $m(A) = \varphi(\mathbf{1}_A)$. Then m is finitely additive ($m(A \cup B) = m(A) + m(B)$, for every disjoint $A, B \subset \mathbb{F}_2$) and left invariant ($m(gA) = m(A)$, for every $g \in \mathbb{F}_2$ and $A \subset \mathbb{F}_2$).

Let a and b be the free generators of $\mathbb{F}_2$. Let S be the set of elements of $\mathbb{F}_2$ whose reduced form begins with a non-zero power of a, and put $T = \mathbb{F}_2 \setminus S$. Then $aT \subset S$, $bS \cup b^2 S \subset T$ and $bS \cap b^2 S = \emptyset$. Thus, we get $m(S) \geq m(aT) = m(T) \geq m(bS \cup b^2 S) = m(bS) + m(b^2 S) = 2m(S)$. This implies that $m(S) = m(T) = 0$. Since $m(S) + m(T) = m(\mathbb{F}_2) = 1$, this provides a contradiction. ∎

Exercise 3.12. Let Γ_1 and Γ_2 be any countable groups such that $|\Gamma_1| > 1$ and $|\Gamma_2| > 2$. Prove that the free product group $\Gamma = \Gamma_1 * \Gamma_2$ is not amenable.

Proposition 3.11 implies that any countable group which contains $\mathbb{F}_2$ as subgroup is non-amenable. Whether the converse is true, i.e., whether *every countable non-amenable group contains* $\mathbb{F}_2$, became known as *von Neumann's problem.* This was settled, in the negative, by Olshanskii in 1980, who proved that every large enough prime p ($p > 10^{75}$), there are *Tarski monster p-groups* (i.e., groups whose only proper non-trivial subgroup is the cyclic group with p elements) which are not amenable.

Nevertheless, the following *measurable* version of von Neumann's problem has a positive answer:

Theorem 3.13 (Gaboriau and Lyons, [GL07])**.** *Let* Γ *be a countable non-amenable group. Then there is a free ergodic p.m.p. action* $\Gamma \curvearrowright (X, \mu)$ *(for instance, the Bernoulli action* $\Gamma \curvearrowright ([0,1]^\Gamma, \mathbf{Leb}^\Gamma)$*) and a free ergodic p.m.p. action* $\mathbb{F}_2 \curvearrowright (X, \mu)$ *such that*

$$\mathcal{R}(\mathbb{F}_2 \curvearrowright X) \subset \mathcal{R}(\Gamma \curvearrowright X).$$

Conversely, if $\mathcal{R}(\mathbb{F}_2 \curvearrowright X) \subset \mathcal{R}(\Gamma \curvearrowright X)$, for some free p.m.p. actions $\mathbb{F}_2, \Gamma \curvearrowright (X, \mu)$, then Γ is non-amenable (see Exercises 3.18 and 3.19).

Exercise 3.14. Let $\mathcal{R}$ be an ergodic p.m.p. equivalence relation. Prove that there exists an ergodic hyperfinite p.m.p. subequivalence relation $\mathcal{S} \subset \mathcal{R}$.

Deduce that if Γ is an infinite countable group, then for every ergodic p.m.p. action $\Gamma \curvearrowright (X,\mu)$, there is an ergodic p.m.p. action $\mathbb{Z} \curvearrowright (X,\mu)$ such that $\mathcal{R}(\mathbb{Z} \curvearrowright X) \subset \mathcal{R}(\Gamma \curvearrowright X)$.

3.2. **Amenable equivalence relations.**

Theorem 3.15 (Ornstein and Weiss, [OW80])**.** *If Γ and Λ are infinite amenable groups, then any ergodic p.m.p. actions of Γ and Λ on non-atomic standard probability spaces are orbit equivalent.*

Definition 3.16. A p.m.p. equivalence relation $\mathcal{R}$ on (X,μ) is *amenable* if there is a sequence of Borel functions $\xi_n : \mathcal{R} \to [0,\infty)$ such that $\xi_n^x : [x]_{\mathcal{R}} \to [0,\infty)$ given by $\xi_n^x(z) = \xi_n(x,z)$ satisfy

(1) $\xi_n^x \in \ell^1([x]_{\mathcal{R}})$ and $\|\xi_n^x\|_1 = 1$, for every $x \in X$, and
(2) $\|\xi_n^x - \xi_n^y\|_1 \to 0$, for all $y \in [x]_{\mathcal{R}}$, for almost every $x \in X$.

Lemma 3.17. *If $\Gamma \curvearrowright (X,\mu)$ is a p.m.p. action of a countable amenable group Γ, then $\mathcal{R}(\Gamma \curvearrowright X)$ is an amenable p.m.p. equivalence relation.*

Proof. By Theorem 3.9, we can find a sequence of non-negative functions $f_n \in \ell^1(\Gamma)$ such that $\|f_n\|_1 = 1$, for all n, and $\lim_{n\to\infty} \|g \cdot f_n - f_n\|_1 = 0$, for all $g \in \Gamma$. Then it is easy to see that the sequence of functions $\xi_n : \mathcal{R}(\Gamma \curvearrowright X) \to [0,\infty)$ given by

$$\xi_n(x,y) = \sum_{g\in\Gamma,\, x=g\cdot y} f_n(g)$$

witnesses the amenability of $\mathcal{R}(\Gamma \curvearrowright X)$. ■

Exercise 3.18. Let $\Gamma \curvearrowright (X,\mu)$ be a free p.m.p. action of a countable group Γ. Prove that if $\mathcal{R}(\Gamma \curvearrowright X)$ is an amenable p.m.p. equivalence relation, then Γ is an amenable group.

Exercise 3.19. Let $\mathcal{S} \subset \mathcal{R}$ be p.m.p. equivalence relations. Prove that if $\mathcal{R}$ is amenable, then $\mathcal{S}$ is amenable.

Lemma 3.20. *Let $\mathcal{R}$ be an amenable p.m.p. equivalence relation on (X,μ). Then $\mathcal{R}|Y$ is an amenable p.m.p. equivalence relation, for every Borel set $Y \subset X$.*

By Lemma 3.17, Theorem 3.15 follows by combining Theorem 2.3 with the next result.

Theorem 3.21 (Connes, Feldman and Weiss, [CFW81])**.** *Any amenable p.m.p. equivalence relation is hyperfinite.*

Theorem 3.21 follows by combining the next three lemmas (3.22,3.25,3.26). We start with the following key lemma:

Lemma 3.22. *Let $\mathcal{R}$ be an amenable p.m.p. equivalence relation on (X,μ). Then for every $\theta_1,...,\theta_k \in [[\mathcal{R}]]$ and $\varepsilon > 0$, there are a non-null Borel set $Y \subset X$ and a finite p.m.p subequivalence relation $\mathcal{S} \subset \mathcal{R}|Y$ such that*

$$\mu(\{x \in Y \cap \mathrm{dom}(\theta_i) \mid \theta_i(x) \notin [x]_{\mathcal{S}}\}) \leq \varepsilon \cdot \mu(Y), \text{ for every } 1 \leq i \leq k.$$

The following fact will be needed in the proof of Lemma 3.22.

Exercise 3.23. Let $\mathcal{R}$ be a countable p.m.p. equivalence relation on (X,μ). Prove that for every Borel function $\xi : \mathcal{R} \to [0,\infty)$ we have that

$$\int_X \Big(\sum_{y\in[x]_{\mathcal{R}}} \xi(x,y)\Big)\mathrm{d}\mu(x) = \int_X \Big(\sum_{x\in[y]_{\mathcal{R}}} \xi(x,y)\Big)\mathrm{d}\mu(y).$$

Proof of Lemma 3.22. For every $1 \le i \le k$, let $\widetilde{\theta}_i \in [\mathcal{R}]$ extending θ_i. Then it is enough to prove the conclusion for $\widetilde{\theta}_1, ..., \widetilde{\theta}_k$ instead of $\theta_1, ..., \theta_k$. Thus, we may assume that $\theta_1, ..., \theta_k \in [\mathcal{R}]$.

Let $\xi_n : \mathcal{R} \to [0, \infty)$ be a sequence of Borel functions as in Definition (3.16). Then $\|\xi_n^{\theta_i(x)} - \xi_n^x\|_1 \to 0$, for all $1 \le i \le k$ and almost every $x \in X$. Since $\|\xi_n^{\theta_i(x)} - \xi_n^x\|_1 \le 2$, for all $x \in X$, the dominated convergence theorem implies that $\int_X \|\xi_n^{\theta_i(x)} - \xi_n^x\|_1 \, \mathrm{d}\mu(x) \to 0$, for every $1 \le i \le k$. Thus, there is $n \ge 1$ such that $\xi := \xi_n$ satisfies

$$\int_X \sum_{i=1}^k \|\xi^{\theta_i(x)} - \xi^x\|_1 \, \mathrm{d}\mu(x) < \varepsilon = \varepsilon \cdot \int_X \|\xi^x\|_1 \, \mathrm{d}\mu(x). \tag{3.2}$$

For $y \in X$, let $\xi_y : [y]_{\mathcal{R}} \to [0, \infty)$ be given by $\xi_y = \xi(x, y)$. By Exercise (3.23), (3.2) rewrites as

$$\int_X \sum_{i=1}^k \|\xi_y \circ \theta_i - \xi_y\|_1 \, \mathrm{d}\mu(y) < \varepsilon \cdot \int_X \|\xi_y\|_1 \, \mathrm{d}\mu(y). \tag{3.3}$$

For $y \in X$ and $t > 0$, let $A_y^t = \{x \in [y]_{\mathcal{R}} \mid \xi^y(x) > t\}$. By Exercise 3.10, (3.3) rewrites as

$$\int_X \int_0^\infty \sum_{i=1}^k |\theta_i^{-1}(A_y^t) \triangle A_y^t| \, \mathrm{d}t \, \mathrm{d}\mu(y) < \varepsilon \cdot \int_X \int_0^\infty |A_y^t| \, \mathrm{d}t \, \mathrm{d}\mu(y). \tag{3.4}$$

By Fubini's theorem we conclude that there is $t > 0$ such that if $A_y := A_y^t$, then the Borel set

$$Z := \{y \in X \mid \sum_{i=1}^k |\theta_i^{-1}(A_y) \triangle A_y| < \varepsilon \cdot |A_y|\}$$

is non-null.

Let $W \subset Z$ be a non-null Borel set such that $A_y \cap A_{y'} = \emptyset$, for all $y \neq y' \in W$. Then $Y := \sqcup_{y \in W} A^y$ is a non-null Borel set. We consider the p.m.p. equivalence relation $\mathcal{S} \subset \mathcal{R}|Y$ whose classes are the sets of the form A_y, for some $y \in W$. Since $|A_y| \le \|\xi_y\|_1 / t$, for all $y \in X$, we get that

$$\int_X |A_y| \, \mathrm{d}\mu(y) \le \big(\int_X \|\xi_y\|_1 \, \mathrm{d}\mu(y) \big)/t = 1/t < \infty,$$

and thus A_y is finite for almost every $y \in X$. This implies that $\mathcal{S}$ is a finite equivalence relation.

Finally, let $1 \le i \le k$. Let $x \in Y$ and $y \in W$ such that $x \in A_y$. If $\theta_i(x) \notin [x]_{\mathcal{S}}$, then $x \notin \theta_i^{-1}(A_y)$. This implies that

$$\mu(\{x \in Y \mid \theta_i(x) \notin [x]_{\mathcal{S}}\}) \le \int_W |A_y \setminus \theta_i^{-1}(A_y)| \, \mathrm{d}\mu(y) \le \varepsilon \cdot \int_W |A_y| \, \mathrm{d}\mu(y) = \varepsilon \cdot \mu(Y),$$

and finishes the proof. ■

Exercise 3.24. Let (X, μ) be a standard probability space, and $\theta : X \to X$ a Borel map such that $\mu(\{x \in X \mid \theta(x) \neq x\}) > 0$. Prove that there is a non-null Borel set $A \subset X$ such that $\theta(A) \cap A = \emptyset$.

Lemma 3.25. *Let $\mathcal{R}$ be an amenable p.m.p. equivalence relation on (X, μ). Then for every $\varphi_1, ..., \varphi_k \in [\mathcal{R}]$ and $\varepsilon > 0$, there is a finite p.m.p subequivalence relation $\mathcal{T} \subset \mathcal{R}$ such that*

$$\mu(\{x \in X \mid \varphi_i(x) \notin [x]_{\mathcal{T}}\}) \le \varepsilon, \textit{ for every } 1 \le i \le k.$$

Proof. We adapt an argument due to Popa (see the proof of [AP19, Theorem 11.1.17]). Let $\varepsilon > 0$. Let $\mathcal{F}$ the be set of all (necessarily countable) families $\{Y_j\}_{j \in J}$ of non-null, pairwise disjoint Borel subsets of X for which we can find finite p.m.p. subequivalence relations $\mathcal{S}_j \subset \mathcal{R}|Y_j$, for every $j \in J$, such that denoting $Z_1 = \sqcup_{j\in J} Y_j$ and $\mathcal{T}_1 = \sqcup_{j\in J}\mathcal{S}_j \subset \mathcal{R}|Z_1$ we have that

$$(3.5)\qquad \mu(\{x \in Z_1 \mid \varphi_i(x) \notin [x]_{\mathcal{T}_1}) + \mu(\{x \notin Z_1 \mid \varphi_i(x) \in Z_1\}) \leq \varepsilon \cdot \mu(Z_1), \text{ for every } 1 \leq i \leq k.$$

Since $\mathcal{R}$ is amenable, Lemma 3.22 provides a non-null Borel set $Y \subset X$ and finite p.m.p. equivalence relation $\mathcal{S} \subset \mathcal{R}|Y$ such that $\mu(\{x \in Y \mid \varphi_i(x) \notin [x]_{\mathcal{S}}\}) \leq \varepsilon/2 \cdot \mu(Y)$, for every $1 \leq i \leq k$. Since

$$\mu(\{x \notin Y \mid \varphi_i(x) \in Y\}) = \mu(\{x \in Y \mid \varphi_i(x) \notin Y\}) \leq \mu(\{x \in Y \mid \varphi_i(x) \notin [x]_{\mathcal{S}}\}) \leq \varepsilon/2 \cdot \mu(Y),$$

we get that $\{Y\} \in \mathcal{F}$, and thus $\mathcal{F}$ is non-empty.

By Zorn's lemma, $\mathcal{F}$ admits a maximal element, $\{Y_j\}_{j\in J}$, with respect to inclusion. In order to finish the proof, it suffices to argue that $Z_1 := \sqcup_{j\in J} Y_j \subset X$ is co-null.

Assuming otherwise, let $Z = X \setminus Z_1$. Let θ_i be the restriction of φ_i to $Z \cap \varphi_i^{-1}(Z)$, for all $1 \leq i \leq k$. Since $\mathcal{R}|Z$ is amenable by Lemma 3.20, by applying Lemma 3.22 to $\theta_1^{\pm}, ..., \theta_k^{\pm} \in [[\mathcal{R}|Z]]$ we find a non-null Borel set $Z_2 \subset Z$ and a finite p.m.p. equivalence relation $\mathcal{T}_2 \subset \mathcal{R}|Z_2$ such that

$$(3.6)\qquad \mu(\{x \in Z_2 \cap \text{dom}(\theta_i) \mid \theta(x) \notin [x]_{\mathcal{T}_2}\}) \leq (\varepsilon/2) \cdot \mu(Z_2), \text{ for every } \theta \in \{\theta_1^{\pm}, ..., \theta_k^{\pm}\}.$$

By applying (3.6) to θ_i we get that

$$(3.7)\qquad \mu(\{x \in Z_2 \mid \varphi_i(x) \in Z \text{ and } \varphi_i(x) \notin [x]_{\mathcal{T}_2}\}) \leq (\varepsilon/2) \cdot \mu(Z_2), \text{ for every } 1 \leq i \leq k.$$

Moreover, we have that

$$\begin{aligned}\theta_i(\{x \in Z \setminus Z_2 \mid \varphi_i(x) \in Z_2\}) &= \{x \in Z_2 \cap \text{dom}(\theta_i^{-1}) \mid \theta_i^{-1}(x) \in Z \setminus Z_2\} \\ &\subset \{x \in Z_2 \cap \text{dom}(\theta_i^{-1}) \mid \theta_i^{-1}(x) \notin [x]_{\mathcal{T}_2}\},\end{aligned}$$

and since θ_i is measure preserving, applying (3.6) to θ_i^{-1} gives that

$$(3.8)\qquad \mu(\{x \in Z \setminus Z_2 \mid \varphi_i(x) \in Z_2\}) \leq (\varepsilon/2) \cdot \mu(Z_2), \text{ for every } 1 \leq i \leq k.$$

Next, we observe that $\{x \in Z_1 \cup Z_2 \mid \varphi_i(x) \notin [x]_{\mathcal{T}_1 \cup \mathcal{T}_2}\}$ is a subset of

$$\{x \in Z_1 \mid \varphi_i(x) \notin [x]_{\mathcal{T}_1}\} \cup \{x \notin Z_1 \mid \varphi_i(x) \in Z_1\} \cup \{x \in Z_2 \mid \varphi_i(x) \in Z \text{ and } \varphi_i(x) \notin [x]_{\mathcal{T}_2}\},$$

and that $\{x \notin Z_1 \cup Z_2 \mid \varphi_i(x) \in Z_1 \cup Z_2\}$ is a subset of

$$\{x \notin Z_1 \mid \varphi_i(x) \in Z_1\} \cup \{x \in Z \setminus Z_2 \mid \varphi_i(x) \in Z_2\}.$$

In combination with (3.7) and (3.8), we derive that for all $1 \leq i \leq k$ we have

$$\mu(\{x \in Z_1 \cup Z_2 \mid \varphi_i(x) \notin [x]_{\mathcal{T}_1 \cup \mathcal{T}_2}\}) + \mu(\{x \notin Z_1 \cup Z_2 \mid \varphi_i(x) \in Z_1 \cup Z_2\}) \leq \varepsilon \cdot \mu(Z_1 \cup Z_2).$$

This contradicts the maximality of $\{Y_j\}_{j\in J}$ and finishes the proof. ■

Lemma 3.26. *A countable p.m.p. equivalence relation $\mathcal{R}$ on (X, μ) is hyperfinite if and only if for every $\varphi_1, ..., \varphi_k \in [\mathcal{R}]$ and $\varepsilon > 0$, we can find a finite subequivalence relation $\mathcal{T} \subset \mathcal{R}$ such that*

$$\mu(\{x \in X \mid \varphi_i(x) \notin [x]_{\mathcal{T}}\}) < \varepsilon, \text{ for every } 1 \leq i \leq k.$$

Proof. The implication ($\Rightarrow$) is obvious. To prove the implication ($\Leftarrow$), we use Theorem 1.10 to write $\mathcal{R} = \mathcal{R}(\Gamma \curvearrowright X)$, for some countable group Γ. Enumerate $\Gamma = \{\varphi_n\}_{n\in\mathbb{N}} \subset [\mathcal{R}]$.

By the hypothesis, for any $k \in \mathbb{N}$, we can find a finite p.m.p. equivalence relation $\mathcal{T}_k \subset \mathcal{R}$ such that

$$\mu(\{x \in X \mid \varphi_i(x) \notin [x]_{\mathcal{T}_k}\}) \leq 1/2^k, \text{ for every } 1 \leq i \leq k.$$

Then $\mathcal{R}_n := \bigcap_{k\geq n} \mathcal{T}_k$, for $n \in \mathbb{N}$, is a finite p.m.p. equivalence relation such that $\mathcal{R}_n \subset \mathcal{R}_{n+1}$ and

$$\mu(\{x \in X \mid \varphi_i(x) \notin [x]_{\mathcal{R}_n}\}) \leq 1/2^{n-1}, \text{ for every } 1 \leq i \leq n.$$

This inequality implies the existence of a co-null Borel set $Y \subset X$ such that for every $i \in \mathbb{N}$, we have $\varphi_i(x) \in [x]_{\cup_{n\in\mathbb{N}}\mathcal{R}_n}$, for all $x \in Y$. Thus, $\mathcal{R}|Y = \cup_{n\in\mathbb{N}}\mathcal{R}_n|Y$, which implies that $\mathcal{R}$ is hyperfinite. ■

4. Orbit equivalence and von Neumann algebras

The theory of orbit equivalence, initiated by H. Dye in [Dy59], was originally motivated by the theory of von Neumann algebras [MvN36,MvN43]. To explain this connection, we will start with a brief introduction to von Neumann algebras, and refer the reader to [AP19,Io19] for more information.

4.1. von Neumann algebras basics.

Definition 4.1. A *Hilbert space* $\mathcal{H}$ is a vector space endowed with a scalar product $\langle\cdot,\cdot\rangle : \mathcal{H}\times\mathcal{H} \to \mathbb{C}$ such that $\langle x, x\rangle = 0 \Rightarrow x = 0$ and $\mathcal{H}$ is complete with respect to the norm $\|x\| = \sqrt{\langle x, x\rangle}$.

Examples 4.2. (of Hilbert spaces):

- $\mathbb{C}^n$, for $n \geq 1$, with the Euclidean scalar product $\langle(x_1, ..., x_n), (y_1, ..., y_n)\rangle = x_1\bar{y}_1+...+x_n\bar{y}_n$.
- $L^2(X)$, for any measure space (X, μ), with the scalar product $\langle f, g\rangle = \int_X f\bar{g}\, \mathrm{d}\mu$.
- $\ell^2(I)$, for any set I, with the scalar product $\langle f, g\rangle = \sum_{i\in I} f(i)\overline{g(i)}$.
- Every Hilbert space $\mathcal{H}$ has an *orthonormal basis*: a set $\{\xi_i\}_{i\in I}$ such that $\langle\xi_i, \xi_j\rangle = \delta_{i,j}$, for all $i, j \in I$, and $\xi = \sum_{i\in I}\langle\xi, \xi_i\rangle\xi_i$, for all $\xi \in \mathcal{H}$. Consequenty, $\mathcal{H}$ is isomorphic to $\ell^2(I)$.

Definition 4.3. Let $\mathcal{H}$ be a Hilbert space. A map $T : \mathcal{H} \to \mathcal{H}$ is called *linear* if it satisfies $T(a\xi + b\eta) = aT(\xi) + bT(\eta)$, for all $a, b \in \mathbb{C}$ and $\xi, \eta \in \mathcal{H}$. A linear map $T : \mathcal{H} \to \mathcal{H}$ is called *bounded* if $\|T\| := \sup\{\|T(\xi)\| \mid \|\xi\| \leq 1\} < \infty$. A linear bounded map $T : \mathcal{H} \to \mathcal{H}$ is called a *bounded (linear) operator*. We denote by $\mathbb{B}(\mathcal{H})$ the algebra of all bounded operators $T : \mathcal{H} \to \mathcal{H}$.

Exercise 4.4. Let $T \in \mathbb{B}(\mathcal{H})$. Prove that there exists a unique $T^* \in \mathbb{B}(\mathcal{H})$, called the *adjoint* of T, such that $\langle T\xi, \eta\rangle = \langle\xi, T^*\eta\rangle$, for all $\xi, \eta \in \mathcal{H}$. Prove that $\|T^*\| = \|T\|$ and $\|T^*T\| = \|T\|^2$.

Definition 4.5. An operator $T \in \mathbb{B}(\mathcal{H})$ is called:

- *self-adjoint* if $T^* = T$.
- a *projection* if $T = T^* = T^2$.
- a *unitary* if $T^*T = TT^* = \mathrm{Id}_{\mathcal{H}}$.
- an *isometry* if $\|T\xi\| = \|\xi\|$, for all $\xi \in \mathcal{H}$, or equivalently $T^*T = \mathrm{Id}_{\mathcal{H}}$.

Remark 4.6. We denote by $\mathcal{U}(\mathcal{H})$ the group unitary operators $T \in \mathbb{B}(\mathcal{H})$. A *unitary representation* of a countable group Γ on a Hilbert space $\mathcal{H}$ is a group homomorphism $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$.

Definition 4.7. We consider the following topologies on $\mathbb{B}(\mathcal{H})$:

- the *norm topology*: $T_i \to T$ iff $\|T_i - T\| \to 0$.
- the *strong operator topology (SOT)*: $T_i \to T$ iff $\|T_i(\xi) - T(\xi)\| \to 0$, for all $\xi \in \mathcal{H}$.
- the *weak operator topology (WOT)*: $T_i \to T$ iff $\langle T_i(\xi), \eta\rangle \to \langle T(\xi), \eta\rangle$, for all $\xi, \eta \in \mathcal{H}$.

Definition 4.8. Let $\mathcal{H}$ be a Hilbert space.

- A subalgebra $A \subset \mathbb{B}(H)$ is called a $*$*-algebra* if $T^* \in A$, for every $T \in A$.
- A $*$-subalgebra $A \subset \mathbb{B}(H)$ is called a *C*-algebra* if it closed in the norm topology.
- A $*$-subalgebra $A \subset \mathbb{B}(H)$ is called a *von Neumann algebra* if it is WOT-closed.

Definition 4.9. A map $\pi : A \to B$ between two C*-algebras is a $*$*-homomorphism* if it is a homomorphism ($\pi(a + b) = \pi(a) + \pi(b)$, $\pi(ab) = \pi(a)\pi(b)$, $\pi(\lambda a) = \lambda\pi(a)$, for all $a, b \in A$, $\lambda \in \mathbb{C}$) and satisfies $\pi(a^*) = \pi(a)^*$ for all $a \in A$. A bijective $*$-homomorphism is called a $*$*-isomorphism*.

Examples 4.10. (of von Neumann algebras):

1. $\mathbb{B}(\mathcal{H})$, in particular $\mathbb{M}_n(\mathbb{C}) = \mathbb{B}(\mathbb{C}^n)$.
2. $L^\infty(X)$, for any standard probability space (X, μ).

(3) Let $B \subset \mathbb{B}(\mathcal{H})$ be a set such that $T^* \in B$, for every $T \in B$. Then the *commutant* of B, defined as $B' = \{T \in \mathbb{B}(\mathcal{H}) \mid TS = ST, \text{ for every } S \in B\}$ is a von Neumann algebra.

Surprisingly, the next theorem shows that every von Neumann algebra arises this way!

Theorem 4.11 (von Neumann's bicommutant theorem)**.** *If $M \subset \mathbb{B}(H)$ is a unital $*$-subalgebra, then the following three conditions are equivalent:*

(1) M is WOT-closed.
(2) M is SOT-closed.
(3) $M = M'' := (M')'$.

This is a beautiful result which asserts that for $*$-algebras, the analytic condition of being closed in the WOT is equivalent to the algebraic condition of being equal to their double commutant.

Proposition 4.12. *Let (X, μ) be a standard probability space. Define $\pi : L^\infty(X) \to \mathbb{B}(L^2(X))$ by letting $\pi_f(\xi) = f\xi$, for all $f \in L^\infty(X)$ and $\xi \in L^2(X)$. Then $\pi(L^\infty(X))' = \pi(L^\infty(X))$. Therefore, $\pi(L^\infty(X)) \subset \mathbb{B}(L^2(X))$ is a maximal abelian von Neumann subalgebra.*

Proof. Let $T \in \pi(L^\infty(X))'$ and put $g = T(1)$. Then $fg = \pi_f T(1) = T\pi_f(1) = T(f)$ and hence

$$\|fg\|_2 = \|T(f)\|_2 \leqslant \|T\| \, \|f\|_2, \quad \text{for every} \quad f \in L^\infty(X).$$

Let $\varepsilon > 0$ and $f = 1_{\{x \in X \mid |g(x)| \geqslant \|T\| + \varepsilon\}}$. Then it is clear that $\|fg\|_2 \geqslant (\|T\| + \varepsilon)\|f\|_2$. In combination with the last inequality, we get that $(\|T\| + \varepsilon)\|f\|_2 \leqslant \|T\|\|f\|_2$, and so $f = 0$, almost everywhere. Thus, we conclude that $g \in L^\infty(X)$. Since $T(f) = fg = \pi_g(f)$, for all $f \in L^\infty(X)$, and $L^\infty(X)$ is $\|.\|_2$-dense in $L^2(X)$, it follows that $T = \pi_g \in L^\infty(X)$. ■

4.2. **The group measure space construction.** Let $\Gamma \curvearrowright (X, \mu)$ be a p.m.p. action of a countable group Γ on a standard probability space (X, μ). Consider the Hilbert space $\mathcal{H} := L^2(X) \otimes \ell^2(\Gamma)$. (If $\mathcal{H}_1, \mathcal{H}_2$ are Hilbert spaces, then tensor Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ is defined as the closure of the algebraic tensor product $\mathcal{H}_1 \otimes_{\text{alg}} \mathcal{H}_2$ w.r.t. scalar product $\langle \xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2 \rangle = \langle \xi_1, \eta_1 \rangle \cdot \langle \xi_2, \eta_2 \rangle$).

We define a unitary representation $\sigma : \Gamma \to \mathcal{U}(L^2(X))$ by $\sigma_g(\xi)(x) = \xi(g^{-1}x)$, for all $\xi \in L^2(X)$. Then $\sigma_g(L^\infty(X)) = L^\infty(X)$, for all $g \in \Gamma$. Further, we define a unitary representation $u : \Gamma \to \mathcal{U}(\mathcal{H})$ and a $*$-homomorphism $\pi : L^\infty(X) \to \mathbb{B}(\mathcal{H})$ by letting

$$u_g(\xi \otimes \delta_h) = \sigma_g(\xi) \otimes \delta_{g^{-1}h} \text{ and } \pi(f)(\xi \otimes \delta_h) = f\xi \otimes \delta_h.$$

We view $L^\infty(X) \subset \mathbb{B}(\mathcal{H})$, via π. Then it is easy to see that

$$u_g f u_g^* = \sigma_g(f), \text{ for all } g \in \Gamma \text{ and every } f \in L^\infty(X).$$

Definition 4.13. The *group measure space von Neumann algebra* $L^\infty(X) \rtimes \Gamma \subset \mathbb{B}(\mathcal{H})$ is defined as the WOT-closure of the linear span of $\{fu_g | f \in L^\infty(X), g \in \Gamma\}$. We denote $M := L^\infty(X) \rtimes \Gamma$.

Proposition 4.14. *The linear functional $\tau : M \to \mathbb{C}$ given by $\tau(x) = \langle x(1 \otimes \delta_e), 1 \otimes \delta_e \rangle$ is*

(1) unital: $\tau(1) = 1$.
(2) positive: $\tau(a^*a) \geq 0$, for every $a \in M$.
(3) faithful: if $\tau(a^*a) = 0$, for some $a \in M$, then $a = 0$.
(4) normal: τ is WOT-continuous on $(M)_1 = \{a \in M \mid \|a\| \leq 1\}$.
(5) a trace: $\tau(ab) = \tau(ba)$, for every $a, b \in M$

Proof. It is clear that τ is unital and normal. Since $\tau(a^*a) = \langle a^*a(1\otimes\delta_e), 1\otimes\delta_e\rangle = \|a(1\otimes\delta_e)\|^2 \geq 0$, it is also positive. To see that τ is faithful, let $a \in M$ such that $\tau(a^*a) = 0$ and thus $a(1\otimes\delta_e) = 0$. Fix $f \in L^\infty(X)$ and $g \in \Gamma$. Define $\rho(f) \in \mathbb{B}(\mathcal{H})$ and $v_g \in \mathcal{U}(\mathcal{H})$ by letting $\rho(f)(\xi\otimes\delta_h) = \xi\sigma_h(f)\otimes\delta_h$ and $v_g(\xi\otimes\delta_h) = \xi\otimes\delta_{hg}$. Then $\rho(f)$, v_g commute with $\pi(f')$, $u_{g'}$, for every $f' \in L^\infty(X)$ and $g' \in \Gamma$. This implies that $\rho(f), v_g \in M'$ and thus

$$a(f\otimes\delta_g) = av_g\rho(f)(1\otimes\delta_e) = v_g\rho(f)a(1\otimes\delta_e) = 0.$$

Since this holds for every $f \in L^\infty(X)$ and $g \in \Gamma$, we get that $a = 0$ and hence τ is faithful.

Finally, note that for all $f \in L^\infty(X)$ and $g \in \Gamma$ we have that

$$\tau(fu_g) = \langle fu_g(1\otimes\delta_e), 1\otimes\delta_e\rangle = \langle f\otimes\delta_g, 1\otimes\delta_e\rangle = \delta_{g,e}\int_X f \,\mathrm{d}\mu.$$

If $f_1, f_2 \in L^\infty(X)$ and $g_1, g_2 \in \Gamma$, then $f_1u_{g_1}f_2u_{g_2} = f_1\sigma_{g_1}(f_2)u_{g_1g_2}$ and $f_2u_{g_2}f_1u_{g_2} = f_2\sigma_{g_2}(f_1)u_{g_2g_1}$. Since $\tau(\sigma_g(f)) = \tau(f)$, for all $f \in L^\infty(X)$ and $g \in \Gamma$, we get that $\tau(f_1u_{g_1}f_2u_{g_2}) = \tau(f_2u_{g_2}f_1u_{g_2})$. This implies that τ is a trace. ∎

Definition 4.15. A von Neumann algebra M is called *tracial* if it admits a unital, positive, faithful, normal trace $\tau : M \to \mathbb{C}$.

Lemma 4.16. *A p.m.p. action $\Gamma \curvearrowright (X,\mu)$ is ergodic iff any function $f \in L^2(X)$ which satisfies that $\sigma_g(f) = f$, for all $g \in \Gamma$, is essentially constant.*

Proof. ($\Leftarrow$) If Y is a Γ-invariant set, then $f = 1_Y \in L^2(X)$ is a Γ-invariant function. Thus, there is $c \in \mathbb{C}$ such that $f = c$. As $f^2 = f$, we get that $c \in \{0,1\}$, hence $\mu(Y) = \int_X f \,\mathrm{d}\mu = c \in \{0,1\}$.

($\Rightarrow$) Let $f \in L^2(X)$ be a Γ-invariant function. If f is not constant, then it admits at least two distinct essential values $z, w \in \mathbb{C}$. Let $\delta = |z-w|/2$. Then $Y = \{x \in X|\ |f(x) - z| < \delta\}$ and $Z = \{x \in X|\ |f(x) - w| < \delta\}$ are disjoint, Γ-invariant, measurable sets. Since $\mu(Y) > 0$ and $\mu(Z) > 0$, we get a contradiction with the ergodicity of the action. ∎

Note that $A := L^\infty(X) \subset M$ is an abelian von Neumann subalgebra.

Proposition 4.17. *The following hold:*

1. *If the action $\Gamma \curvearrowright (X,\mu)$ is free, then $A \subset M$ is* maximal abelian, *i.e., $A' \cap M = A$.*
2. *If the action $\Gamma \curvearrowright (X,\mu)$ is free and ergodic, then M is a* factor, *i.e., $\mathcal{Z}(M) := M' \cap M = \mathbb{C}1$.*

Proof. (1) Assume that the action is free. Let $a \in A' \cap M$ and write $a(1\otimes\delta_e) = \sum_{g\in\Gamma} a_g\otimes\delta_g$, where $a_g \in L^2(X)$. Fix $f \in L^\infty(X)$ and define $\rho(f) \in \mathbb{B}(\mathcal{H})$ by letting $\rho(f)(\xi\otimes\delta_h) = \xi\sigma_h(f)\otimes\delta_h$. Then $\rho(f) \in M'$ and thus we have

$$\sum_{g\in\Gamma} fa_g\otimes\delta_g = \pi(f)a(1\otimes\delta_e) = a\pi(f)(1\otimes\delta_e) = a(f\otimes\delta_e) = a\rho(f)(1\otimes\delta_e) = \rho(f)a(1\otimes\delta_e) = \sum_{g\in\Gamma}\sigma_g(f)a_g\otimes\delta_g$$

Hence, $fa_g = \sigma_g(f)a_g$, for all $f \in L^\infty(X)$ and $g \in \Gamma$. Put $Y_g = \{x \in X|a_g(x) \neq 0\}$, for $g \neq e$. From the last equality we get that $f(g^{-1}x) = f(x)$, for almost every $x \in Y_g$, for all $f \in L^\infty(X)$. Since (X,μ) is a standard probability space, we can find a sequence of measurable sets $X_n \subset X$, which separate points in X. By applying the last identity to $f = 1_{X_n}$, for all $n \geq 1$, we deduce that $g^{-1}x = x$, for almost every $x \in Y_g$. Since the action is free, we get that $\mu(Y_g) = 0$, hence $a_g = 0$. Since this holds for all $g \in \Gamma\setminus\{e\}$, we conclude that $a \in A$.

(2) Since the action is free, (1) implies that $\mathcal{Z}(M) = \{a \in A \mid \sigma_g(a) = a, \forall g \in \Gamma\}$. Since the action is also ergodic, the conclusion follows from Lemma 4.16. ∎

4.3. **Cartan subalgebras and orbit equivalence.** If M is a tracial von Neumann algebra, then a von Neumann subalgebra $A \subset M$ is called a *Cartan subalgebra* if the following conditions hold:

(1) A is maximal abelian, i.e., $A' \cap M = A$, and
(2) the linear span of the normalising group $\mathcal{N}_M(A) = \{u \in \mathcal{U}(M) \mid uAu^* = A\}$ is WOT-dense in M.

By Proposition 4.17 (1), $L^\infty(X) \subset L^\infty(X) \rtimes \Gamma$ is a Cartan subalgebra, for any free p.m.p. action $\Gamma \curvearrowright (X, \mu)$. It is a fundamental observation of Singer (1955) that the isomorphism class of the Cartan inclusion $L^\infty(X) \subset L^\infty(X) \rtimes \Gamma$ captures exactly the orbit equivalence class of the action $\Gamma \curvearrowright (X, \mu)$.

Proposition 4.18 (Singer, [Si55])**.** *If $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ are free p.m.p. actions, then the following conditions are equivalent:*

(1) There exists a $*$-isomorphism $\pi : L^\infty(X) \rtimes \Gamma \to L^\infty(Y) \rtimes \Lambda$ such that $\pi(L^\infty(X)) = L^\infty(Y)$.
(2) The actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ are orbit equivalent.

Proof. We will only sketch the proof of the implication (2) $\Rightarrow$ (1) and refer the reader to [AP19, Chapter 12] and [Io19, Section 12] for the reverse implication.

Denote $A = L^\infty(X), B = L^\infty(Y), M = L^\infty(X) \rtimes \Gamma$ and $N = L^\infty(Y) \rtimes \Lambda$. Let $\theta : (X, \mu) \to (Y, \nu)$ be a measure space isomorphism such that $\theta(\Gamma \cdot x) = \Lambda \cdot \theta(x)$, for almost every $x \in X$. Define a $*$-isomorphism $\pi : A \to B$ by letting $\pi(a) = a \circ \theta^{-1}$. Our goal is to show that π extends to a $*$-isomorphism $\pi : M \to N$.

To this end, fix $g \in \Gamma$. Then $(\theta \circ g^{-1} \circ \theta^{-1})(y) \in \Lambda \cdot y$, for almost every $y \in Y$. For $h \in \Lambda$, put $Y_{g,h} = \{y \in Y \mid (\theta \circ g^{-1} \circ \theta^{-1})(y) = h^{-1} \cdot y\}$. Then $\{Y_{g,h}\}_{h\in\Lambda}$ is a measurable partition of Y. Since $h^{-1} \cdot Y_{g,h} = \{y \in Y \mid (\theta \circ g \circ \theta^{-1})(y) = h \cdot y\}$, we also have that $\{h^{-1} \cdot Y_{g,h}\}_{h\in\Lambda}$ is a measurable partition of Y. Using these facts, one checks that $\pi(u_g) = \sum_{h\in\Lambda} 1_{Y_{g,h}} u_h$ defines a unitary in N.

We define $U : L^2(X) \otimes \ell^2(\Gamma) \to L^2(Y) \otimes \ell^2(\Lambda)$ by letting

$$U(a \otimes \delta_g) = \sum_{h\in\Lambda} (a \circ \theta^{-1}) 1_{Y_{g,h}} \otimes \delta_h, \text{ for every } g \in \Gamma.$$

Then U gives a well-defined unitary operator. We leave it as an exercise to check that

$$U(au_g\,(b \otimes \delta_h)) = \pi(a)\pi(u_g)\,U(b \otimes \delta_h), \text{ for every } a, b \in A \text{ and } g, h \in \Gamma.$$

Thus, $Uau_gU^* = \pi(a)\pi(u_g)$, for all $a \in A$ and $g \in \Gamma$. Since the linear span of $\{au_g \mid a \in A, g \in \Gamma\}$ is WOT-dense in M, we deduce that $UMU^* \subset N$. Hence, the $*$-isomorphism $\pi : A \to B$ extends to a $*$-homomorphism $\pi : M \to N$ given by $\pi(T) = UTU^*$. We leave it as exercise to check that $\pi(M) = N$, and thus π is the desired $*$-isomorphism. ∎

5. Non-orbit equivalent actions: free groups of different ranks

Theorem 5.1 (Gaboriau, [Ga00, Ga02])**.** *If $2 \leq m \neq n \leq \infty$, then any two free ergodic p.m.p. actions $\mathbb{F}_m \curvearrowright (X, \mu)$ and $\mathbb{F}_n \curvearrowright (Y, \nu)$ are* not *orbit equivalent.*

The original proof of this theorem [Ga00] uses the notion of *cost* of p.m.p. equivalence relations introduced by Levitt and developed extensively by Gaboriau (see Definition 5.2). In [Ga02], Gaboriau gave a new proof of this result based on his notion of *ℓ^2-Betti numbers* of p.m.p. equivalence relations. Here we reproduce a proof of this theorem presented in [AP19, Section 18.3] due to Vaes, itself a version in the spirit of operator algebras of a previous proof by Gaboriau. This relies on showing that the cost of the orbit equivalence relation of any free ergodic p.m.p. action of $\mathbb{F}_m$ is exactly m.

Definition 5.2. Let $\mathcal{R}$ be a countable p.m.p. equivalence relation on (X,μ). A *graphing* for $\mathcal{R}$ is a countable collection $\mathcal{G}=(\varphi_k)\subset[[\mathcal{R}]]$ such that $\mathcal{R}$ is the smallest equivalence relation containing $(x,\varphi_k(x))$, for every $x\in\text{dom}(\varphi_k)$ and k. The *cost* of a graphing $\mathcal{G}$ of $\mathcal{R}$ is defined by $\text{cost}(\mathcal{R}):=\sum_k\mu(\text{dom}(\varphi_k))$. The *cost* of $\mathcal{R}$ is defined by $\text{cost}(\mathcal{R}):=\inf\{\text{cost}(\mathcal{G})\mid\mathcal{G}\text{ graphing of }\mathcal{R}\}$.

Theorem 5.3 (Gaboriau, [Ga00])**.** *Let $\Gamma\curvearrowright(X,\mu)$ be a free ergodic p.m.p. action of $\Gamma=\mathbb{F}_m$. Then the cost of the orbit equivalence relation $\mathcal{R}:=\mathcal{R}(\Gamma\curvearrowright X)$ is equal to m.*

Proof of Theorem 5.3. Let $a_1,...,a_m$ be free generators of Γ. Then $(a_i)_{i=1}^m$ is a graphing of $\mathcal{R}$ of cost m. Thus, we have to show that if $\mathcal{G}=(\varphi_k)$ is any graphing of $\mathcal{R}$, then $\text{cost}(\mathcal{G})\geq m$. By restricting φ_k if necessary, we may assume that for every k, there is $g_k\in\Gamma$ such that $\varphi_k(x)=g_k\cdot x$, for every $x\in\text{dom}(\varphi_k)$. Moreover, we may assume that there are only finitely many φ_k's, say $\varphi_1,...,\varphi_N$. (This requires an argument, which we leave as an exercise).

Denote $M=L^\infty(X)\rtimes\Gamma$ and let $(u_g)_{g\in\Gamma}\subset\mathcal{U}(M)$ be the canonical unitaries. Let $a_1,...,a_m\in\Gamma$ be free generators. We consider the space of 1-cocycles

$$Z^1(\Gamma,M)=\{c:\Gamma\to M\mid c(gh)=u_gc(h)+c(g),\text{ for all }g,h\in\Gamma\}.$$

Notice that $Z^1(\Gamma,M)$ is a right M-module. Since every cocycle $c:\Gamma\to M$ is uniquely determined the by the values $c(a_1),...,c(a_m)$ we have an isomorphism of right M-modules $\Phi:M^{\otimes m}\to Z^1(\Gamma,M)$ given by $\Phi(x_1,...,x_m)$ is the unique cocycle $c:\Gamma\to M$ such that $c(a_1)=x_1,...,c(a_m)=x_m$.

For every k, let $p_k=1_{g_kA_k}$ and note that $p_k\in L^\infty(X)\subset M$ is a projection.

We define a right M-modular map $\Psi:Z^1(\Gamma,M)\to\oplus_kp_kM$ by letting $\Psi(c)=\oplus_kp_kc(g_k)$. We claim that Ψ is injective. To prove the claim, let $c:\Gamma\to M$ be a cocycle such that $p_kc(g_k)=0$, for all k. Then for every $k_1,..,k_n$ we have that $(p_{k_l}\circ(g_{k_1}...g_{k_{l-1}})^{-1})u_{g_{k_1}}...u_{g_{k_{l-1}}}c(g_{k_l})=0$ and thus

$$\begin{aligned}&p_{k_1}(p_{k_2}\circ g_{k_1}^{-1})...(p_{k_n}\circ(g_{k_1}...g_{k_{n-1}})^{-1})c(g_{k_1}...g_{k_n})\\ &=\sum_{l=1}^n[p_{k_1}(p_{k_2}\circ g_{k_1}^{-1})...(p_{k_n}\circ(g_{k_1}...g_{k_{n-1}})^{-1})]u_{g_{k_1}}...u_{g_{k_{l-1}}}c(g_{k_l})=0\end{aligned}\tag{5.1}$$

We are now ready to show that if $g\in\Gamma$, then $c(g)=0$. Let $A\subset X$ be a non-null measurable set. Then we can find a non-null measurable subset $B\subset A$ and $k_1,..,k_n$ such that B is contained in the domain of $\varphi_{k_1}\circ...\circ\varphi_{k_n}$ and $g\cdot x=(\varphi_{k_1}\circ...\circ\varphi_{k_n})(x)$, for all $x\in B$. Equivalently $1_B\leq(1_{A_{k_1}}\circ(g_{k_2}...g_{k_n}))...(1_{A_{k_{n-1}}}\circ g_{k_n})1_{A_{k_n}}$ and $g=g_{k_1}...g_{k_n}$. Thus, we have

$$\begin{aligned}1_{gB}=1_{g_{k_1}...g_{k_n}B}&=1_B\circ(g_{k_1}...g_{k_n})^{-1}\\ &\leq(1_{A_{k_1}}\circ g_{k_1}^{-1})...(1_{A_{k_n}}\circ(g_{k_1}...g_{k_n})^{-1})\\ &=p_{k_1}...(p_{k_n}\circ(g_{k_1}...g_{k_{n-1}})^{-1}).\end{aligned}$$

This and (5.1) imply that $1_{gB}c(g)=1_{gB}c(g_1...g_k)=1_{gB}p_{k_1}...(p_{k_n}\circ(g_{k_1}...g_{k_{n-1}})^{-1})c(g_1...g_k)=0$. Thus, for every non-null set A we can find a non-null subset B such that $1_{gB}c(g)=0$. This implies that $c(g)=0$. Since $g\in\Gamma$ is arbitrary, we derive that $c=0$ and thus Ψ is injective.

Therefore, $\Psi\circ\Phi:M^{\oplus m}\to\oplus_kp_kM$ is a right M-modular injective map. By using Theorem 5.4 below, we conclude that $m\leq\sum_k\tau(p_k)=\sum_k\mu(A_k)$. ■

Theorem 5.4. *Let (M,τ) be a tracial von Neumann and $\{p_i\}_{i\in I}$, $\{q_j\}_{j\in J}$ be projections in M. Assume that there exists a right M-modular linear injective map $T:\oplus_ip_iM\to\oplus_jq_jM$.*

Then $\sum_{i\in I}\tau(p_i)\leq\sum_{j\in J}\tau(q_j)$.

For a proof of this theorem, see [Io19, Theorem 10.21].

Proof of Theorem 5.1. If two free ergodic p.m.p. actions $\mathbb{F}_m \curvearrowright (X,\mu)$ and $\mathbb{F}_n \curvearrowright (Y,\nu)$ are orbit equivalent, then the orbit equivalence relations $\mathcal{R}(\mathbb{F}_m \curvearrowright X)$ and $\mathcal{R}(\mathbb{F}_n \curvearrowright Y)$ are isomorphic. Thus, they have the same cost, and therefore $m = n$ by Theorem 5.3. ■

6. Non-orbit equivalent actions of non-amenable groups (incomplete section)

6.1. **Overview.** By Ornstein and Weiss' Theorem 3.15, any infinite amenable group Γ has only one free ergodic p.m.p. action up to orbit equivalence. On the other hand, Connes and Weiss [CW80] showed (by using Schmidt's notion of strong ergodicity [Sc80]) that any non-amenable group Γ without Kazhdan's property (T) has at least two actions up to orbit equivalence. Moreover, in the period 1980-2004, many groups were shown to have uncountably many such actions:

Theorem 6.1. *The following countable groups admit uncountably many non-orbit equivalent free ergodic p.m.p. actions:*

(1) Bezuglyi-Golodets, [BG81]*: McDuff's (1969) continuum of groups.*
(2) Gefter-Golodets, [GG89]*: higher rank lattices.*
(3) Hjorth, [Hj02]*: any infinite group with Kazhdan's property (T)*
(4) Monod-Shalom, [MS02]*:* $\mathbb{F}_m \times \mathbb{F}_n$*, for any* $m, n \geq 2$*; more generally, any direct product of non-elementarily hyperbolic groups.*
(5) Gaboriau-Popa, [GP03]*:* $\mathbb{F}_m$*, for any* $m \geq 2$*.*
(6) Popa, [Po04]*: any group admitting an infinite normal subgroup with relative property (T).*
(7) Ioana, [Io04]*: any direct product* $\Gamma_1 \times \Gamma_2$*, with* Γ_1 *infinite amenable and* Γ_2 *non-amenable.*

From these, we highlight the following key advance:

Theorem 6.2 (Gaboriau-Popa, [GP03])**.** *The free group* $\mathbb{F}_m$ *has uncountably many pairwise non-orbit equivalent free ergodic p.m.p. actions, for any* $m \geq 2$*.*

The proof of Theorem 6.2 uses a separability argument in combination with Popa's influential notion of *rigid* actions, which we discuss in the next section. To outline the strategy of the proof, assume that $\mathbb{F}_3 = \langle a,b,c\rangle \curvearrowright^{\alpha} (X,\mu)$ is a free ergodic p.m.p. action whose restriction to $\mathbb{F}_2 = \langle a,b\rangle$ is ergodic and rigid and restriction to $\mathbb{Z} = \langle c\rangle$ is ergodic. Examples of such actions are provided by the restriction of the natural action $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright (\mathbb{T}^2,\lambda^2)$ to any subgroup isomorphic to $\mathbb{F}_3$.

Dye's theorem gives a free p.m.p. action $\oplus_{\mathbb{N}}\mathbb{Z}/2\mathbb{Z} \curvearrowright (X,\mu)$ with $\mathcal{R}(\mathbb{Z} \curvearrowright X) = \mathcal{R}(\oplus_{\mathbb{N}}\mathbb{Z}/2\mathbb{Z} \curvearrowright X)$. If $I \subset \mathbb{N}$, by applying Dye's theorem again, we can find a free p.m.p. action $\mathbb{Z} \curvearrowright^{\beta_I} (X,\mu)$ such that $\mathcal{R}(\mathbb{Z} \curvearrowright^{\beta_I} X) = \mathcal{R}(\oplus_I\mathbb{Z}/2\mathbb{Z} \curvearrowright X)$. Define a free ergodic p.m.p. action $\mathbb{F}_3 \curvearrowright^{\alpha_I} (X,\mu)$ by letting $\alpha_{I|\mathbb{F}_2} = \alpha$ and $\alpha_{I|\mathbb{Z}} = \beta_I$. Then $\mathcal{R}(\mathbb{F}_3 \curvearrowright^{\alpha_I} X) \subset \mathcal{R}(\mathbb{F}_3 \curvearrowright^{\alpha_J} X) \subset \mathcal{R}(\mathbb{F}_3 \curvearrowright^{\alpha} X)$, for all $I \subset J \subset \mathbb{N}$. Using that $\alpha_{|\mathbb{F}_2}$ is rigid, Gaboriau and Popa prove that for any infinite set $I \subset \mathbb{N}$, the set $\{J \subset \mathbb{N} \mid \alpha_J \text{ is orbit equivalent to } \alpha_I\}$ is countable. Thus, uncountably many of the actions $\{\alpha_I \mid I \subset \mathbb{N}\}$ are pairwise not orbit equivalent.

6.2. **Property (T).** Let Γ be a countable group and Γ_0 be a subgroup of Γ.

Definition 6.3. A unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ of Γ on a Hilbert space $\mathcal{H}$ is said to have *almost invariant vectors* if there is a sequence of vectors $\xi_n \in \mathcal{H}$ such that $\|\xi_n\| = 1$, for every n, and $\|\pi(g)\xi_n - \xi_n\| \to 0$, for every $g \in \Gamma$.

Definition 6.4. The group Γ has *property (T)* (of Kazhdan) if for any unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ with almost invariant vectors there is a non-zero $\pi(\Gamma)$-invariant vector.

The inclusion $\Gamma_0 < \Gamma$ (or pair (Γ, Γ_0)) has *relative property (T)* (of Kazhdan-Margulis) if for any unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ with almost invariant vectors there is a non-zero $\pi(\Gamma_0)$-invariant vector.

Example 6.5. $\mathrm{SL}_n(\mathbb{Z})$, for $n \geq 3$, has property (T) [Ka67], and the inclusion $\mathbb{Z}^2 < \mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ has relative property (T) [Ka67, Ma82]. Here, $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{Z}^2$ by matrix multiplication.

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